

Lecture 1A – Electrostatics

Introduction. A brief history of electrostatics. Vectors. The vector dot product. The vector cross product. Area vectors. Coulomb's Law. The electric field. Potential difference. Current density and Ohm's law. Surface integrals. Flux and flux density. Gauss' Law for electrostatics.

Introduction

Electrodynamics – the behaviour of moving “charges” – forms the fundamental basis of electrical engineering. We see the effects of electrodynamics daily: lightning and static electricity; magnets and compasses; all the benefits of power systems including domestic, commercial and industrial lighting, heating and the running of motors; telecommunications networks such as radio, television, telephones and the Internet; and the now ubiquitous computer technology. The applications of electrodynamics are diverse (some are simple devices, others are complex systems), but all are described by a few basic principles – it is these *fundamentals laws* that we will study.

Electrodynamics is everywhere

Historically, electric and magnetic phenomena were studied separately. It was only during the 19th century, through the work of several great physicists such as Oersted and Faraday, that a link was found between the two phenomena. Faraday was the consummate experimentalist with a visionary's sense of the unity of nature. He was the first to conceptualize the “electromagnetic field” – a force field that permeates all of space and which gives rise to both electric and magnetic phenomena.

Faraday visualized fields and discovered electromagnetic induction

It then took the genius of James Maxwell to formulate a set of consistent and harmonious mathematical relations between electric and magnetic fields – a unified field theory – which predicted electromagnetic waves and led to the formulation of relativity in the early 20th century by Einstein.

These mathematical relations, now called “Maxwell's equations”, successfully describe *all* large-scale electromagnetic phenomena – charged rods, currents in circuits, rotating machines, the way that light propagates through a vacuum, etc. Maxwell's equations are the second most successful equations discovered so far (after the equations of quantum mechanics) in terms of experimental

Maxwell's equations form the basis for all electromagnetic phenomena

1A.2

verification. The equations are used all around us. We live in a world dominated by them – from power generation and the machines that drive industry, to the miniature electronics that has spawned the communication and computing revolution – and so the study of electric and magnetic “fields” is essential for electrical engineers.

Before we embark on a study of electrodynamics, we will firstly consider the much simpler case of *electrostatics*, i.e. the study of electric fields due to static (non-moving) charges.

In retrospect, it is interesting to note that the mathematical equations of *static electric fields*, placed in the framework of the Special Theory of Relativity, also lead to Maxwell’s equations. In this subject, we will follow the historical approach and become familiar with the laws which were postulated based on experimental evidence. The laws in this form are still of great practical use.

A Brief History of Electrostatics

A brief history of
electrostatics

By 600 BCE the ancient Greeks knew that amber (Greek: *elektron*), when rubbed, would attract small quantities of straw, silk, and other light objects. Nothing further was done with this knowledge. Nothing further was learned about electricity for 2200 years.

During the 17th century, there was a lot of attention paid to terrestrial magnetism – because of navigation – and very little to electricity. Scientists were too preoccupied with mechanics and optics (e.g. Newton).

In the 18th century, experiments on frictional electricity became numerous, and the *art* of performing electrical demonstrations developed rapidly. What was still lacking however, was quantitative knowledge of the forces acting between charged bodies. Coulomb provided this in 1785.

Vectors

The description of electric (and magnetic) phenomena using mathematical equations requires the use of vectors. Vectors portray both the magnitude and direction of a quantity, and are mathematical entities that possess several very important properties. We will revise some of these properties, and introduce new ones as the need arises.

The Vector Dot Product

The "dot" product of two vectors is defined by:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$$

(1A.1) Dot product defined

For example, when a force moves an object through a small distance $d\mathbf{l}$ it does a small amount of mechanical work dW which is given by:

$$dW = \mathbf{F} \cdot d\mathbf{l}$$

(1A.2) and used in the definition of work

We define the dot product of two vectors this way because the "cos" factor occurs numerous times in the mathematical expressions that describe nature. For example, consider moving a box along the floor:

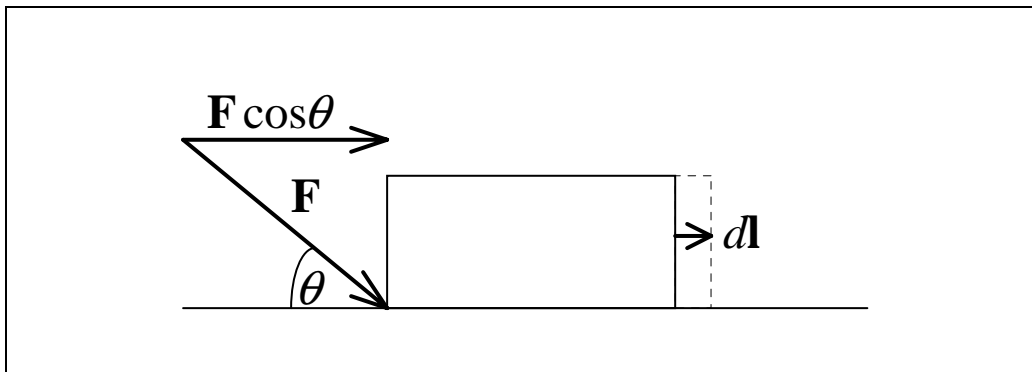


Figure 1A.1

To obtain the work done in moving the box, we first have to figure out the component of the applied force that actually does something useful. That is, we have to find the force that acts in the direction of movement. This "useful" force is shown in the diagram, and you can see that it involves a "cos" term. We can therefore use the shorthand notation of the dot product when writing the expression for the work.

Dot product as a shorthand notation

1A.4

Concept of
differentials

Why did we use differentials in Eq. (1A.2)? When something moves, it generally does not follow a straight line. But if we consider very small displacements, (so small that each displacement is a straight line), then Eq. (1A.2) applies. For example, consider pushing an object through an unusual path:

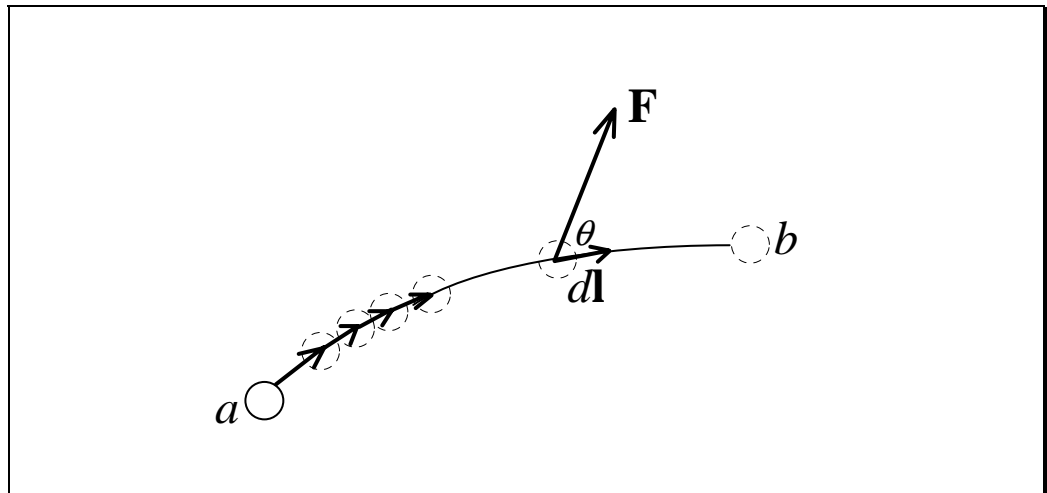


Figure 1A.2

Concept of
differential work
done

An object, when acted upon by a force, is not obliged to move in that direction, as shown in the diagram. We use Eq. (1A.2) to calculate, for each small displacement $d\mathbf{l}$, the small amount of work done in moving the object, dW .

To calculate the total work done, in moving the object from point a to point b , we perform what is known as a "line" integral (we add up the differentials along a certain curve – in this case the curve ab):

Total work done as
an integral

$$W = \int_a^b dW = \int_a^b \mathbf{F} \cdot d\mathbf{l} \quad (1A.3)$$

The Vector Cross Product

The "cross" product of two vectors is defined as:

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{c}}$$

(1A.4) Cross product defined

where the unit vector $\hat{\mathbf{c}}$ has a direction perpendicular to both \mathbf{a} and \mathbf{b} . Once again, the definition of the cross product is based upon its utility and frequency in describing natural phenomena.

To determine the direction of $\hat{\mathbf{c}}$, we use the *Right Hand Screw Rule*. To apply this rule, you *imagine* the vectors \mathbf{a} and \mathbf{b} positioned on a plane. Then "grab hold of" vector \mathbf{a} and rotate it into vector \mathbf{b} so that you mimic screwing a lid on a jar, or tightening a right hand screw. The direction of advance of the lid or screw gives the direction of $\hat{\mathbf{c}}$. This should all happen in your mind, *do not* use your hands to perform this mental operation.

Right Hand Screw Rule defined

The cross product defines a Cartesian coordinate system:

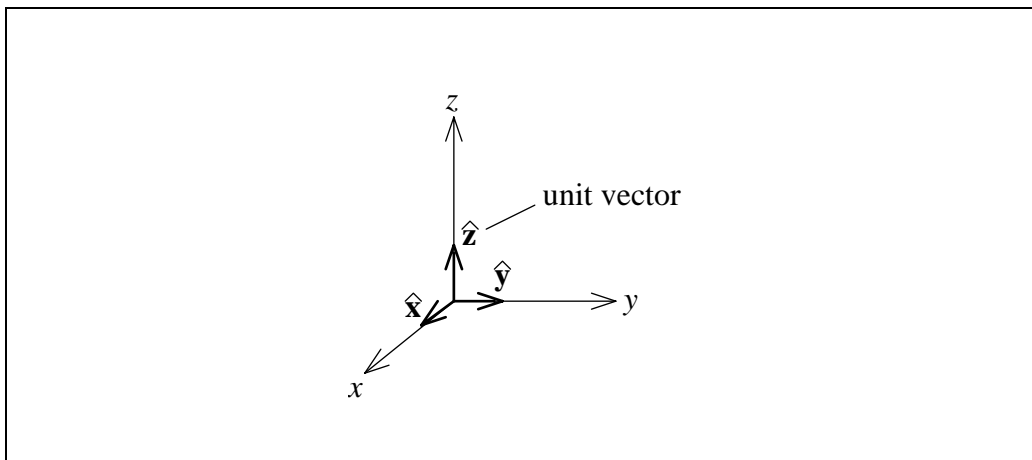


Figure 1A.3

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}}$$

(1A.5) Cartesian coordinate system

1A.6

Area Vectors

Area vector defined

An *area* vector has the job of specifying the size and direction of an area. Direction of an area? Yes, by convention, the *direction* of an area is defined to be the direction perpendicular to the plane of the area. Of course, this implies that the area is flat, but it also applies to curved surfaces that are infinitesimally small.

For example, the area vector for a rectangle would look like:

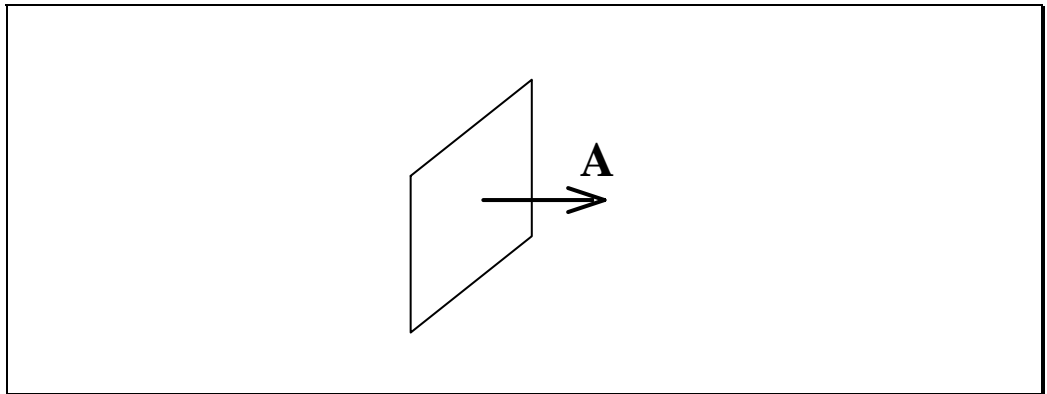


Figure 1A.4

where the *magnitude* of \mathbf{A} would equal the area of the rectangle. An infinitesimally small area, such as part of a sphere would be represented by:

Differential area vector

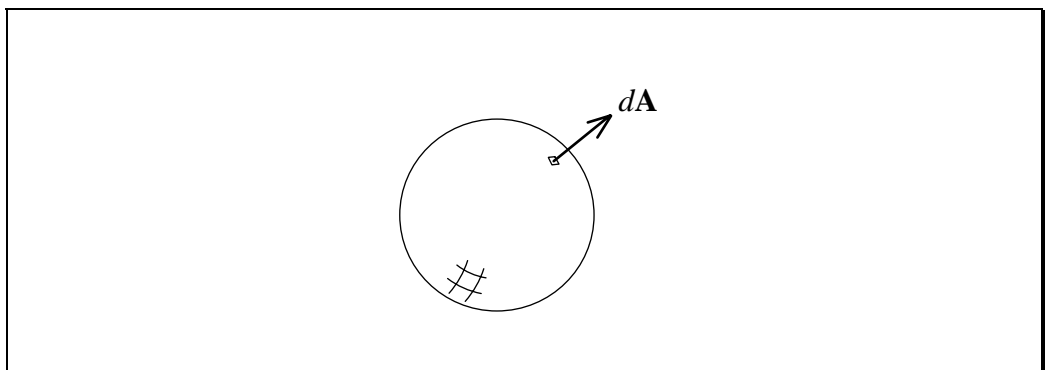


Figure 1A.5

Coulomb's Law

Demo

Rub acrylic on rabbits fur. Use long pith ball. Show attraction, then repulsion. Explain in terms of the transfer of "charge". Talk about action-at-a-distance. Rub ebonite. Show attraction and repulsion. Use short pith ball. Show attraction and no repulsion. Postulate the existence of two types of charge. This is exactly what Benjamin Franklin did, and he labelled them as positive and negative.

We can "measure" how much charge there is by using an electroscope. Describe the operation of the electroscope. Demonstrate the effect of induction of static charges, and how we can deposit either type of charge on the electroscope – touch the electroscope with the acrylic rod. Touching with ebonite rod should neutralize it. Induce a charge with the ebonite rod, earth electroscope, and remove rod. Introduce ebonite rod.

Theory

Charles-Augustin de Coulomb won a prize in 1784 by providing the best method of constructing a ship's compass to the French Academy of Sciences. It was while investigating this problem that Coulomb invented his torsion balance. He showed that charge was distributed on the surface of a conductor, and recognized this as a consequence of the mutual repulsion of like charges according to an inverse square law.

Coulomb's
apparatus

1A.8

In 1785, Coulomb used an apparatus based upon his torsion balance to measure the tiny electrostatic forces caused by two charged spheres. The quantitative results are embodied in Coulomb's Law:

Coulomb's Law

$$\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \hat{\mathbf{R}} \quad \text{N} \quad (1A.6)$$

The quantities expressed in this law are shown below:

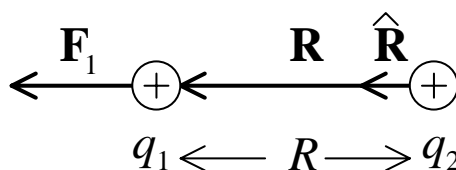


Figure 1A.6

The subscript 1 on the force means this is the force on charge q_1 due to charge q_2 . Therefore, the unit vector $\hat{\mathbf{R}}$ points in a direction that goes from q_2 to q_1 .

Mathematical notation

Here we should clarify some notation. The *vector* \mathbf{R} points from the source of the force (q_2) to the point where the force is felt (q_1). This will *always* be the case throughout this subject. \mathbf{R} points from the source to the effect. The magnitude of the vector \mathbf{R} is just R . The unit vector $\hat{\mathbf{R}}$ has the same properties as \mathbf{R} except its magnitude is one.

The Electric Field

Concept of a "field"

The charge q_1 will feel a force even though nothing is touching it! We know that it is caused by q_2 . We can now imagine some sort of field of influence radiating out from charge q_2 into all of space (3 dimensions). As far as q_1 is concerned, it just finds itself immersed in some sort of space where a force is felt (think of a rocket in deep space and gravity). We can then imagine that something permeates the space even before we place our charge q_1 in it. We

call this a *field*. When we place the charge q_1 in the field, we see a reaction – in this case a force.

With this thinking, it appears that a field exists due solely to q_2 . We call this field the *electric field*, and for a point charge it is defined as:

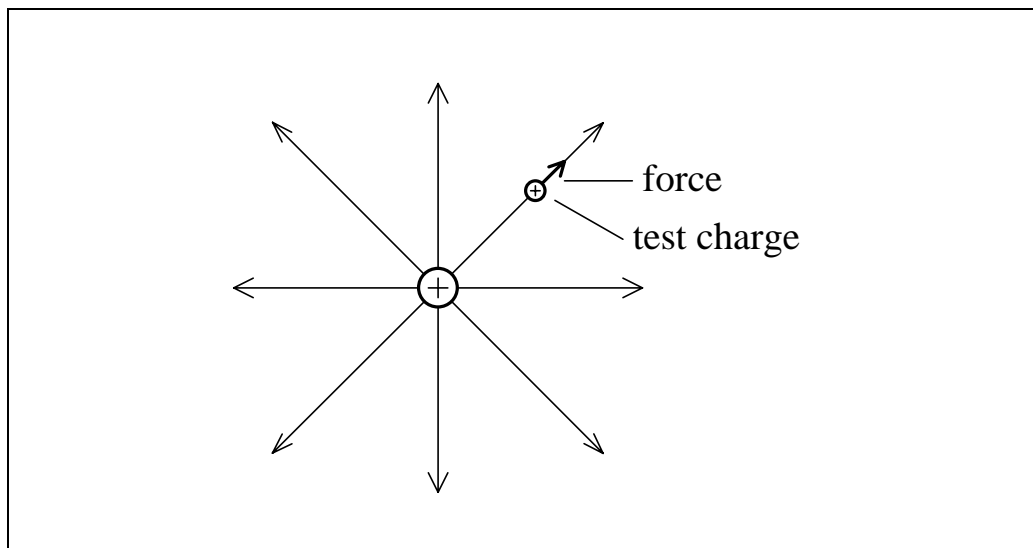
$$\mathbf{E}_1 = \frac{q_2}{4\pi\epsilon_0 R^2} \hat{\mathbf{R}} \quad \text{Vm}^{-1}$$

(1A.7) Electric field of a point charge

Now consider an isolated point charge. The electric field must exist all around it, but let's examine what happens in a two dimensional cross section. We can take a very small positive test charge and place it in the field near the point charge. We will constrain the test charge to move infinitesimally slowly away from the point charge. The path the test charge traces out is called a line of force.

Electric lines of force

For an isolated positive point charge, the lines of force radiate from the charge in all directions:



Lines of force give a "picture" of the field

Figure 1A.7

The lines of force drawn in this manner create a picture of the electric field. The direction of the electric field at any point is given by the direction of the force on the positive test charge.

1A.10

Using these ideas, we can calculate the force on a charge using Coulomb's law:

$$\mathbf{F}_1 = q_1 \mathbf{E}_1 = q_1 \frac{q_2}{4\pi\epsilon_0 R^2} \hat{\mathbf{R}} \quad (1A.8)$$

Computer Demo

Demonstrate field around single isolated charge. It is 3D. The direction at any point is given by the tangent to the line of force if more than one charge is involved. Demonstrate with a +ve and -ve charge.

Demonstrate the effect of not having a small test charge - it distorts the field.

Superposition

Superposition applied to components of a force

In mechanics we often split up a total force on an object into a number of components. Conversely, we can add up a number of components to get the total force. We can do the same with the electrostatic force. Consider the following arrangement:

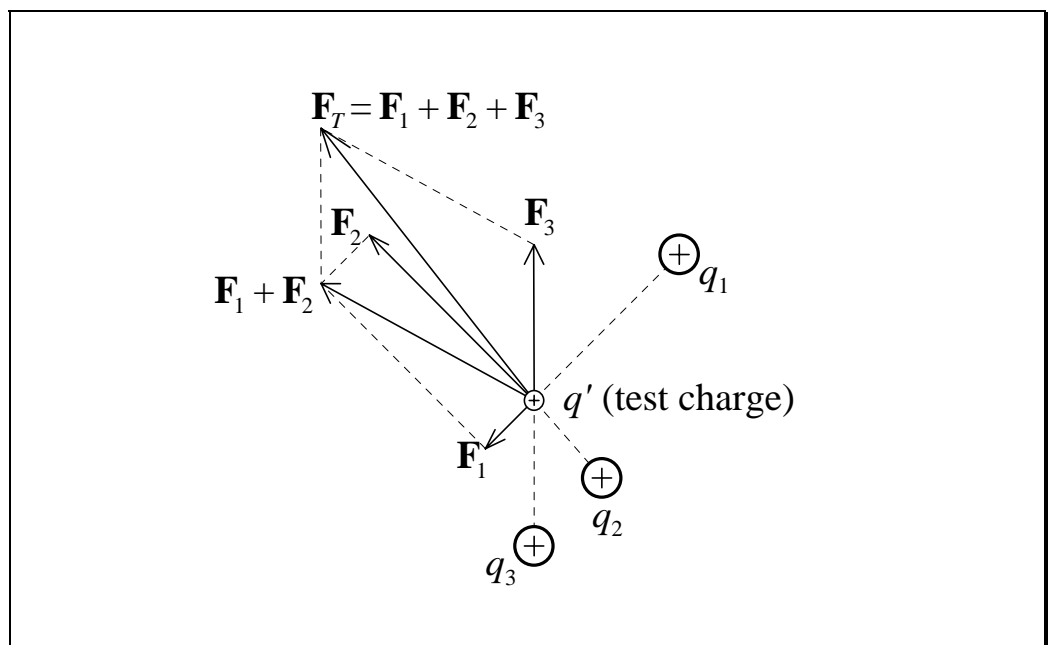


Figure 1A.8

The resultant force on the test charge is given by:

$$\mathbf{F}_T = \sum_{i=1}^3 \mathbf{F}_i = q' \sum_{i=1}^3 \mathbf{E}_i = q' \mathbf{E} \quad (1A.9)$$

Using superposition to calculate the field due to more than one charge

The constant of proportionality in Coulomb's Law, namely $1/4\pi\epsilon_0$, is only true for a vacuum. In any other medium, we generalise Coulomb's Law by replacing ϵ_0 by ϵ :

$$\epsilon = \text{permittivity (or dielectric constant) of the medium} \quad (1A.10a)$$

$$\epsilon_0 = \text{permittivity of free space} = 8.85419 \times 10^{-12} \text{ Fm}^{-1} \quad (1A.10b)$$

Permittivity as an electric property of the medium

For media other than free space, we define relative permittivity:

$$\epsilon_r = \epsilon / \epsilon_0 \quad (1A.11)$$

Relative permittivity defined

1A.12

Potential Difference

Potential difference defined using mechanical work done

Potential difference involves calculating the work done in moving a test charge between two points. Consider the field from a single isolated charge:

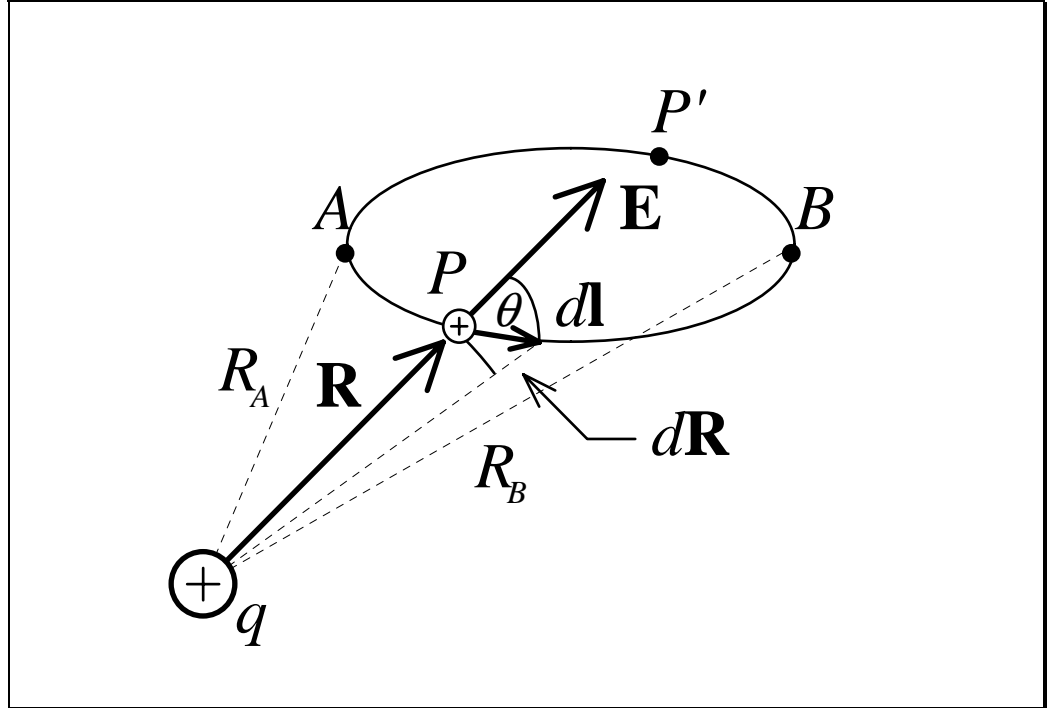


Figure 1A.9

The amount of work we have to do to move a charge by a small increment is given by:

Work done depends on the electric field

$$\begin{aligned} dW &= \mathbf{F}_M \cdot d\mathbf{l} \\ &= -\mathbf{F}_E \cdot d\mathbf{l} \\ &= -q\mathbf{E} \cdot d\mathbf{l} \end{aligned} \quad (1A.12)$$

To move a charge infinitesimally slowly in an electric field, we apply a force that exactly counteracts the Coulomb force. A larger force than the Coulomb force will accelerate the charge. The above equation gives the work done when moving an infinitesimal displacement $d\mathbf{l}$. To find the work done in moving a charge from A to B , just integrate:

Work done for an arbitrary path in an electric field

$$W_{BA} = -q \int_A^B \mathbf{E} \cdot d\mathbf{l} \quad \text{J} \quad (1A.13)$$

Potential difference is the mechanical work done per unit charge.

$$V_{BA} = -\int_A^B \mathbf{E} \cdot d\mathbf{l} \quad \text{V}$$

(1A.14) Potential difference defined

For the case of a single isolated charge, we calculate the integral as follows:

$$\begin{aligned} V_{BA} &= -\int_A^B E \cos \theta dl = -\int_{R_A}^{R_B} E dR \\ &= -\int_{R_A}^{R_B} \frac{q}{4\pi\epsilon R^2} dR = \frac{q}{4\pi\epsilon} \left(\frac{1}{R_B} - \frac{1}{R_A} \right) \end{aligned} \quad (1A.15)$$

For an isolated point charge, we define “absolute potential” at any point to be:

$$V = \frac{q}{4\pi\epsilon R}$$

(1A.16) Absolute potential defined for a point charge

To find the potential difference at two points, we can subtract the absolute potential of one point with the other. *Compare this with Eq. (1A.15).*

The electrostatic \mathbf{E} field is a conservative field. This means that no work is done in moving a charge around a path and back to its starting position – energy is conserved. In Figure 1A.9, along $APBP'A$ we have:

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = 0$$

(1A.17) The electrostatic \mathbf{E} field is conservative

1A.14

Current Density and Ohm's Law

Consider a conducting sheet, conductivity σ , resistivity ρ , cross-sectional area A :

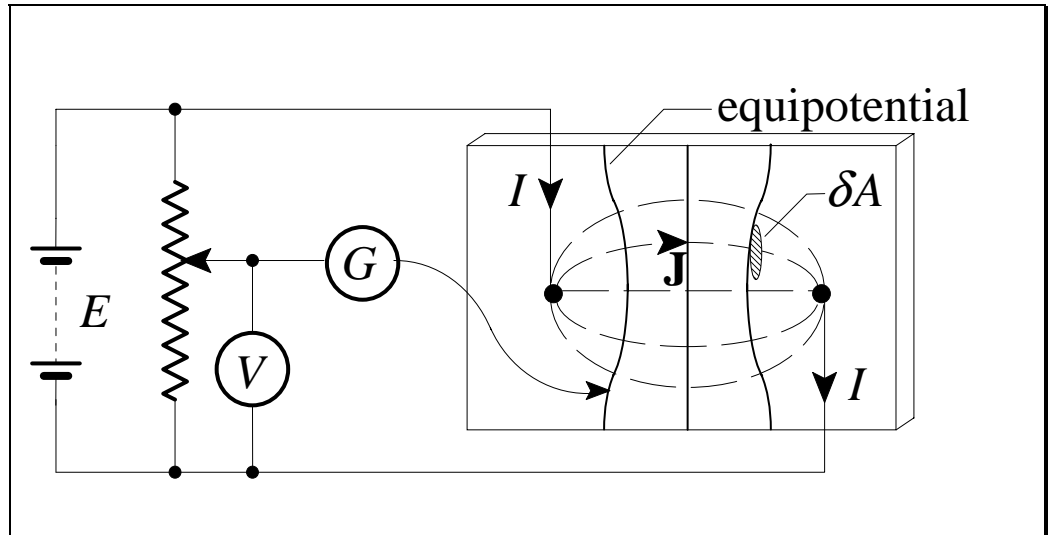


Figure 1A.10

The battery E sets up an \mathbf{E} field, which in turn causes the free charges in the metal sheet to flow along the lines of \mathbf{E} (lines of force). The current density in the metal sheet is defined as:

Current density
defined

$$\mathbf{J} = \lim_{\delta A \rightarrow 0} \frac{\delta I}{\delta A} = \frac{dI}{dA} \text{ and tangent to the } \mathbf{E} \text{ lines} \quad (1A.18)$$

Since \mathbf{J} and \mathbf{E} point in the same direction, they must differ in magnitude only:

Field version of
Ohm's Law

$$\mathbf{J} = \sigma \mathbf{E} \quad (1A.19)$$

Show that this is Ohm's Law.

The equipotentials on the metal sheet are determined using the galvanometer G . *How?* They are always perpendicular to the lines of \mathbf{J} .

Surface Integrals

Consider a uniform current in a conductor. \mathbf{J} and \mathbf{A} are both vectors, and:

$$I = \mathbf{J} \cdot \mathbf{A} \quad (1A.20)$$

Getting current from
a uniform current
density

Does \mathbf{A} have to be the cross section of the conductor for this expression to be true? *Test this out by drawing an area that is not perpendicular to the conductor cross section.*

What do we do if the current density is not uniform? We divide the area up into regions where the current density is uniform and summate over the whole area.

Eventually we come to the surface integral:

$$I = \int_A \mathbf{J} \cdot d\mathbf{A}$$

(1A.21) Getting current from
a current density

1A.16

Flux and Flux Density

Flux as a tool for describing action-at-a-distance

The action-at-a-distance that we see with electrostatics can be explained by postulating a *flux*, ψ , that exerts influence over objects nearby. It does not flow, but emanates, or *streams*, from the source (an electric charge). It permeates all of space. How do we measure the flux that is *streaming* through space?

Think back to how we measured the current going through a conductor. There we defined a current density at each point in the conductor. For electrostatics, if we define an electric flux density at each point in space, and call it \mathbf{D} , then:

Electric flux density defined

$$\mathbf{D} = \lim_{\delta A \rightarrow 0} \frac{\delta \psi}{\delta A} = \frac{d\psi}{dA} \quad (1A.22)$$

When \mathbf{D} is not uniform, to find the electric flux ψ streaming through an area A , we have to perform the integral:

Getting flux from flux density

$$\psi = \int_A \mathbf{D} \cdot d\mathbf{A} \quad (1A.23)$$

Show that this integral has the same value in free space for all surfaces A having the same perimeter.

Lets make the area a closed area and surround some charge. This gives the closed surface integral:

Getting the flux streaming out of a closed surface

$$\psi = \oint_A \mathbf{D} \cdot d\mathbf{A} \quad (1A.24)$$

Show this pictorially for a single charge and a sphere. Obviously, if the charges are the source of flux, then we should get more flux if there is more charge – in our model the amount of flux must be proportional to the amount of charge. This leads to Gauss' Law.

Gauss' Law for Electrostatics

Gauss' Law can be derived from Coulomb's Law, but it is very complicated. It is simpler to give an intuitive definition:

$$\psi = \oint_A \mathbf{D} \cdot d\mathbf{A} = q$$

(1A.25) Gauss' Law
postulated

where q is the charge enclosed by the area A . In words, it says that the total flux streaming through a closed surface is equal to the amount of charge enclosed by that surface. It *does not* say that no flux can stream out of the enclosing surface – it just means that if some does, then it inevitably must stream back into some other part of the surface.

Apply Gauss' Law to the point charge and show that:

$$\mathbf{D} = \epsilon \mathbf{E}$$

(1A.26) Electric flux density
is related to electric
field intensity

This is true in general and relates electric *flux density* to electric *field intensity*.

1A.18

Summary

- The electrostatic force between two infinitesimally small electric charges is

given by Coulomb's Law: $\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \hat{\mathbf{R}} \text{ N}.$

- An infinitesimally small electric charge produces an electric field given by:

$$\mathbf{E}_1 = \frac{q_2}{4\pi\epsilon_0 R^2} \hat{\mathbf{R}} \text{ Vm}^{-1}.$$

- Permittivity, ϵ , is an electric property of a medium.

- The potential difference between two points A and B is given by:

$$V_{BA} = -\int_A^B \mathbf{E} \cdot d\mathbf{l} \text{ V}.$$

- The electrostatic \mathbf{E} field is conservative: $\oint_l \mathbf{E} \cdot d\mathbf{l} = 0.$

- Current density is defined as: $\mathbf{J} = \frac{dI}{dA}.$

- Ohm's Law is given by: $\mathbf{J} = \sigma \mathbf{E}.$

- Electric flux density is defined as: $\mathbf{D} = \frac{d\psi}{dA}.$

- Gauss' Law for electrostatics is: $\oint_A \mathbf{D} \cdot d\mathbf{A} = q.$

- The electric flux density is related to the electric field intensity by: $\mathbf{D} = \epsilon \mathbf{E}.$

References

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Shadowitz, Albert: *The Electromagnetic Field*, Dover Publications, Inc., New York, 1975.

Shamos, Morris H. (Ed.): *Great Experiments in Physics - Firsthand Accounts from Galileo to Einstein*, Dover Publications, Inc., New York, 1959.

Problems

1.

Use Gauss' Law to obtain the electrostatic flux density \mathbf{D} and hence the field intensity \mathbf{E} , at a distance r , in a vacuum, from:

- (a) the centre of a uniformly charged spherical shell, with radius a , and a total charge q , when $r \geq a$.
- (b) as (a) but with $r < a$.
- (c) a line charge with uniform charge density $\lambda \text{ Cm}^{-1}$.
- (d) a plane with uniform charge density $\sigma \text{ Cm}^{-2}$.

2.

Use the above results to derive expressions for the potential difference between two points at radial distances r_a and r_b , if $r_a > r_b$. Draw the field pattern for each case, i.e. lines of force and equipotentials.

3.

Derive an expression for the capacitance per unit length of a coaxial cable. The diameters of the inner and outer conductors are d_1 and d_2 respectively. The insulating material between the conductors has relative permittivity ϵ_r .

4.

A spherical cloud of charge of radius R carries total charge Q . The charge is distributed so that its density is spherically symmetric, i.e. it is a function of the radial distance from the centre of the sphere.

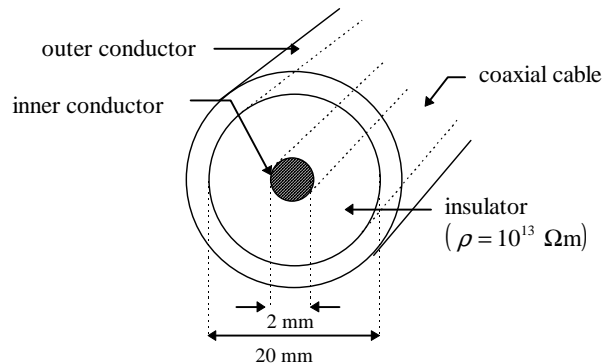
Explain why the “charge cloud” is equivalent to a point charge of Q Coulombs at the centre of the sphere.

Determine the force experienced by an electron, charge $-e$, orbiting the sphere at distance $d \text{ m}$ ($d > R$) from its centre, with constant velocity v .

5.

Explain the four diagrams of “Faraday’s ice pail experiment”.

6.



The voltage across the insulation layer is 100 kV. Determine the leakage current, for 1 km of cable length, flowing from the inner to the outer conductor.

7.

An earthing electrode consists of a metal hemisphere (radius a and zero resistivity) just below the surface of the earth (radius ∞ , resistivity ρ).

Let $a = 0.5$ m, $\rho = 100$ Ωm, fault current through electrode = 1 kA.

- (a) Show that the resistance to earth (at ∞) is $R = \rho/2\pi a$.
- (b) Determine the resistance between two such electrodes very far apart (i.e. first electrode to ∞ then to second electrode).
- (c) Calculate the maximum potential difference between two probes driven into the ground, 0.5 m apart, when the mean distance between the probes and electrode is 100 m.

1A.22

8.

A straight rod AB lies along the x -axis and has a uniformly distributed charge density $\lambda \text{ Cm}^{-1}$.

Show that the x and y components of the \mathbf{E} field at point P are given by:

$$\mathbf{E}_{Px} = \frac{\lambda}{4\pi\epsilon_0 b} (\sin \theta_B - \sin \theta_A) \hat{\mathbf{x}}$$

$$\mathbf{E}_{Py} = \frac{\lambda}{4\pi\epsilon_0 b} (\cos \theta_A - \cos \theta_B) \hat{\mathbf{y}}$$

where: P is a point in the first quadrant, b = distance of P from x -axis.

θ_A & θ_B are the angles AP and BP make with the x -axis.

Also show that for a semi- ∞ line charge (A at origin, B at ∞ , $\theta_A = \pi/2$):

$$E_{Py} = -E_{Px} = \frac{q}{4\pi\epsilon_0 b}$$

$$\mathbf{E}_P = \frac{q}{2\sqrt{2}\pi\epsilon_0 b} \angle 135^\circ$$

and for an ∞ line (A at $-\infty$, B at ∞):

$$\mathbf{E}_{Py} = \mathbf{E}_P = \frac{\lambda}{2\pi\epsilon_0 b} \hat{\mathbf{y}}$$

$$\mathbf{E}_{Px} = \mathbf{0}, \text{ field cylindrical}$$

9.

Derive an expression for the capacitance between two spherical, concentric, metal electrodes (radii R_1 and R_2). The dielectric medium is air.