Lecture 3B – Field Mapping

The method of curvilinear squares. The coaxial cable. The two conductor transmission line.

The Method of Curvilinear Squares

There are various methods we can employ to map out a field. The method of curvilinear squares is based upon the plotting of lines of force and equipotentials, just like our original picture of fields. It is done by hand, and may be iterative. It is used to get an idea of what the field “looks” like and to get estimates of capacitance and inductance of mathematically difficult systems.

One method of field plotting is the method of curvilinear squares

You can conceivably obtain a field "plot" of a three dimensional (3D) field if you are prepared to model in 3D. e.g. construction of a 3D grid with wires representing lines of force and equipotentials.

On paper (the most convenient material) we are restricted to two dimensions (2D), so this method is normally based on 2D problems.

Field plotting is mainly used for 2D problems

Consider a 3D arrangement of conductors that have uniform cross-section, and are infinitely long. There are no field components in the longitudinal direction. (*Why?*) We only have to analyse the field by taking a cross-section. We have seen this before: the infinitely long conductor, the coaxial cable.

2D plots may be applied to 3D problems in certain cases

Consider the electrostatic field around a point charge:

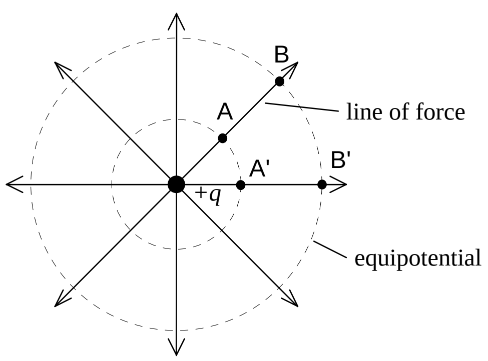


Figure 3B.1

The electric field at A or A', distance R_A from the charge is:

$$\mathbf{E}_A = \frac{q}{4\pi\varepsilon_0 R_A^2}\hat{\mathbf{R}} \tag{3B.1}$$

The absolute potential around a point charge (revisited)

The potential (with respect to infinity) is:

$$\begin{aligned} V_A &= -\int_{\infty}^{R_A} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{R_A} -Edl \\ &= -\int_{\infty}^{R_A} EdR = \frac{-q}{4\pi\varepsilon_0}\int_{\infty}^{R_A}\frac{dR}{R^2} = \frac{q}{4\pi\varepsilon_0 R_A} \end{aligned} \tag{3B.2}$$

to illustrate the concept of “equipotential”

The potential is independent of where the point A lies on the circle. It is only dependent on the distance from the charge. Hence the circle with radius R_A is an equipotential.

Equipotentials and lines of force are always at right angles in electrostatics

Equipotentials are always at right angles to lines of force. Imagine a test charge being moved perpendicular to the direction of the field at all times. Then:

$$\int_A^B \mathbf{E}\cdot d\mathbf{l} = \int_A^B E\cos\theta dl = \int_A^B E\cos 90° dl = 0 \tag{3B.3}$$

The surface of a metal with a static charge is an equipotential, since the tangential part of **E** is zero on the surface. (If **E** were not zero, then charges would redistribute themselves on the surface until there was no force on them – a condition which means the tangential part of **E** is zero).

A field plot is a plot of equipotentials and lines of force

We can now consider a field plot to be composed of two families of lines: one representing lines of force (or equivalently, lines defining tubes of flux); the other representing equipotentials. We will always know where to draw some of the equipotentials: at the surface of conductors.

The field around a point charge (drawn in Figure 3B.1) can be considered as a cross-section of the field around an infinitely long line charge, as far as the field plot is concerned (the previous equations do not apply of course).

In Figure 3B.1, the element:

A curvilinear square shown pictorially

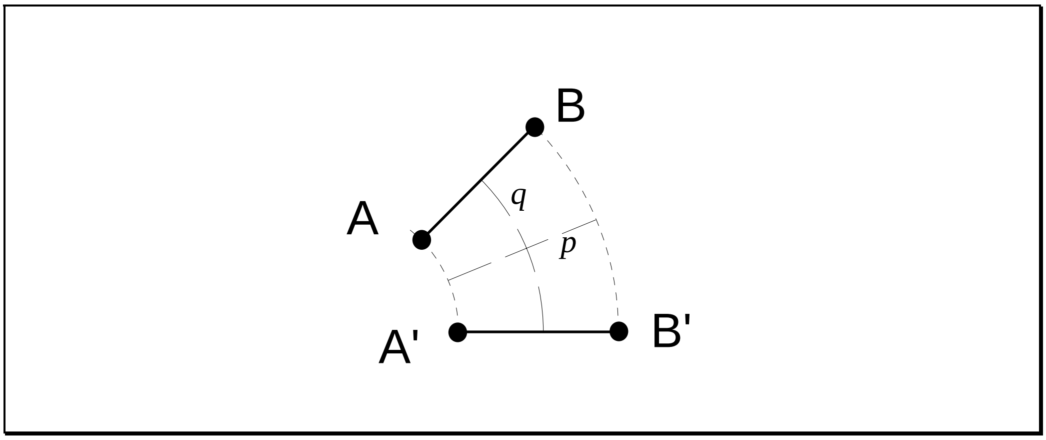


Figure 3B.2

is called a *curvilinear square* if $p = q$. A curvilinear square is a shape with four sides that tends to yield true squares as it is subdivided into smaller and smaller areas by successive halving of the equipotential interval and the flux per tube.

and defined mathematically

We can draw field lines to satisfy the requirement that the density of lines is proportional to the field. We can then draw in equipotentials to obtain curvilinear squares. We can also *not* obtain curvilinear squares, which means the field lines are wrong. The whole process starts again by modifying the field lines to obtain curvilinear squares (if the plot is done in pencil). In other words, we proceed in an iterative fashion (if we knew what the field looked like to begin with, there would be no need to use this method, would there?).

Drawing curvilinear squares is a “trial and error” method

This method of field plotting is very useful for irregular shapes and arrangements of conductors.

Consider the electric field shown below:

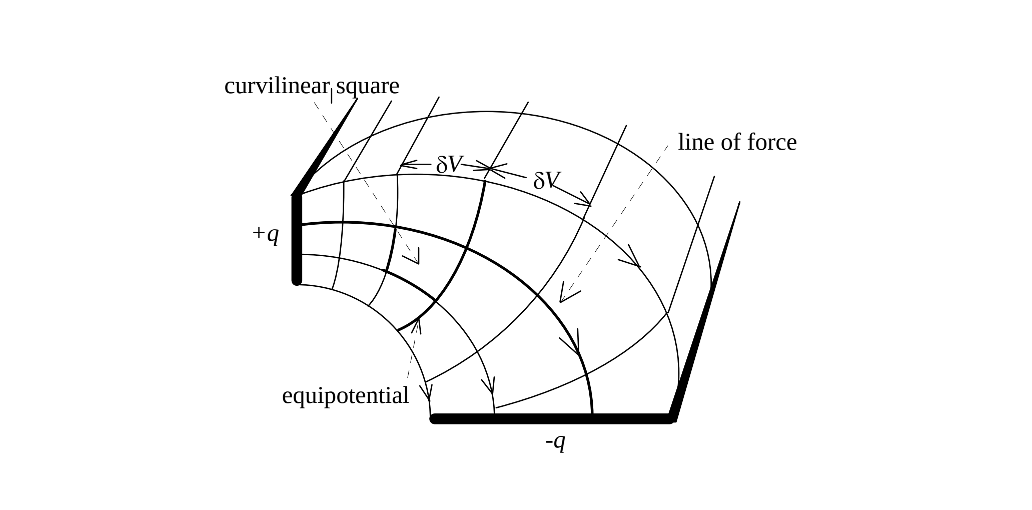


Figure 3B.3

The potential difference between the two conductors is V volts. The LHS conductor has a distributed charge $+q$ and the RHS has $-q$. This is a bad plot. *Why*? Because the last equipotential converges onto another equipotential. The plot will have to be corrected. *Correct the above field plot. Hint: the field lines are wrong too.*

Why do a field plot? One reason is to obtain an estimate of capacitance (per unit length)

Once we get the plot visually right (the curvilinear requirement is met), we may wish to determine the capacitance per unit length between the two conductors, using the field plot.

We know that the capacitance between two conductors is given by:

$$C = \frac{q}{V} \tag{3B.4}$$

We also know from Gauss' Law around one of the conductors:

$$q = \psi \tag{3B.5}$$

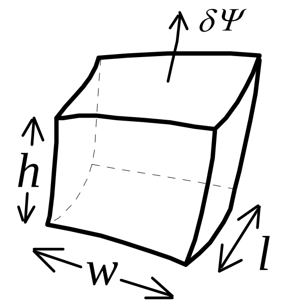
where ψ is the flux emanating from the conductor.

We could then say:

$$C = \frac{\psi}{V} \tag{3B.6}$$

Capacitance defined in terms of flux and potential

To calculate capacitance using this formula, we should first consider an isolated curvilinear cube:



A curvilinear cube has flux streaming *through* it, and potential *across* it

Figure 3B.4

It has a small amount of flux streaming through it, and a small voltage across it. It therefore contributes to the capacitance in some way. If the curvilinear cube is very small, then the flux density **D** may be assumed uniform across the face of the cube so that:

A curvilinear cube is a small capacitor

$$\begin{aligned} \delta\Psi &\approx \mathbf{D} \cdot \delta\mathbf{A} \\ &= \varepsilon E w l \end{aligned} \tag{3B.7}$$

The flux streaming through a curvilinear cube

We can approximate the electric field magnitude E by calculating the small potential that exists across the curvilinear cube:

$$\begin{aligned} \delta V &= -\int_l \mathbf{E} \cdot d\mathbf{l} \\ &= Eh \end{aligned}$$

and the potential across it

$$E = \frac{\delta V}{h} \tag{3B.8}$$

Therefore, the amount of flux streaming through the cube may be expressed as:

$$\delta\Psi \approx \varepsilon \frac{\delta V}{h} wl \tag{3B.9}$$

We can think of each curvilinear cube:

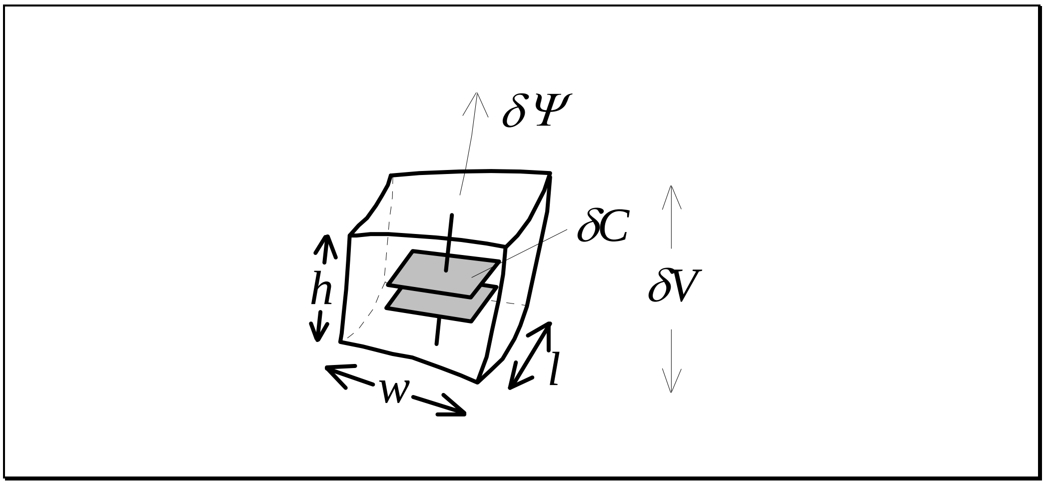


Figure 3B.5

as a small *field cell* whose capacitance is given by:

$$\delta C = \frac{\delta\Psi}{\delta V} = \frac{\varepsilon wl}{h} \text{ F} \tag{3B.10}$$

Also, if the curvilinear cube is small, $w \approx h$ and the flux is given by:

$$\delta\Psi \approx \varepsilon \delta V l \tag{3B.11}$$

The total amount of flux streaming from one of the conductors is obtained by adding up all the small amounts of flux streaming through each flux tube:

The total flux streams through all the curvilinear cubes that are in parallel

$$\psi = \sum_{n_p} \delta\Psi \tag{3B.12}$$

where n_p is the number of flux tubes in parallel (number of curvilinear squares in *parallel*).

The total potential between the two conductors is obtained by adding up all the small amounts of potential between each equipotential, in going from one conductor to the other:

$$V = \sum_{n_s} \delta V \tag{3B.13}$$

The total voltage is across all the curvilinear cubes that are in series

where n_s is the number of equipotentials minus one (number of curvilinear squares in *series*).

We can now determine the capacitance of the structure in this way:

$$C = \frac{\psi}{V} = \frac{\sum_{n_p} \delta\Psi}{\sum_{n_s} \delta V} = \frac{\sum_{n_p} \varepsilon \delta V l}{\sum_{n_s} \delta V} \tag{3B.14}$$

The capacitance using curvilinear squares

But since δV is the same value for each curvilinear square, we have:

$$\begin{aligned} \sum_{n_p} \varepsilon \delta V l &= n_p \varepsilon \delta V l \\ \sum_{n_s} \delta V &= n_s \delta V \end{aligned} \tag{3B.15}$$

We can now define the capacitance per unit length of the two conductors. This is all we can calculate, since the capacitance of infinitely long conductors is infinite. Our answer may be applied to very long conductors with a small error.

$$\boxed{\frac{C}{l} = \varepsilon \frac{n_p}{n_s} \ \text{Fm}^{-1}} \tag{3B.16}$$

The capacitance per unit length using curvilinear squares

Example – Parallel plate capacitor in a uniform dielectric

For a good mental image and for the sake of completeness, we will show the entire field although we realise that due to the symmetry of the arrangement, we could get away with plotting only a $1/4$ of the field:

A field around a parallel plate capacitor showing the fringing field

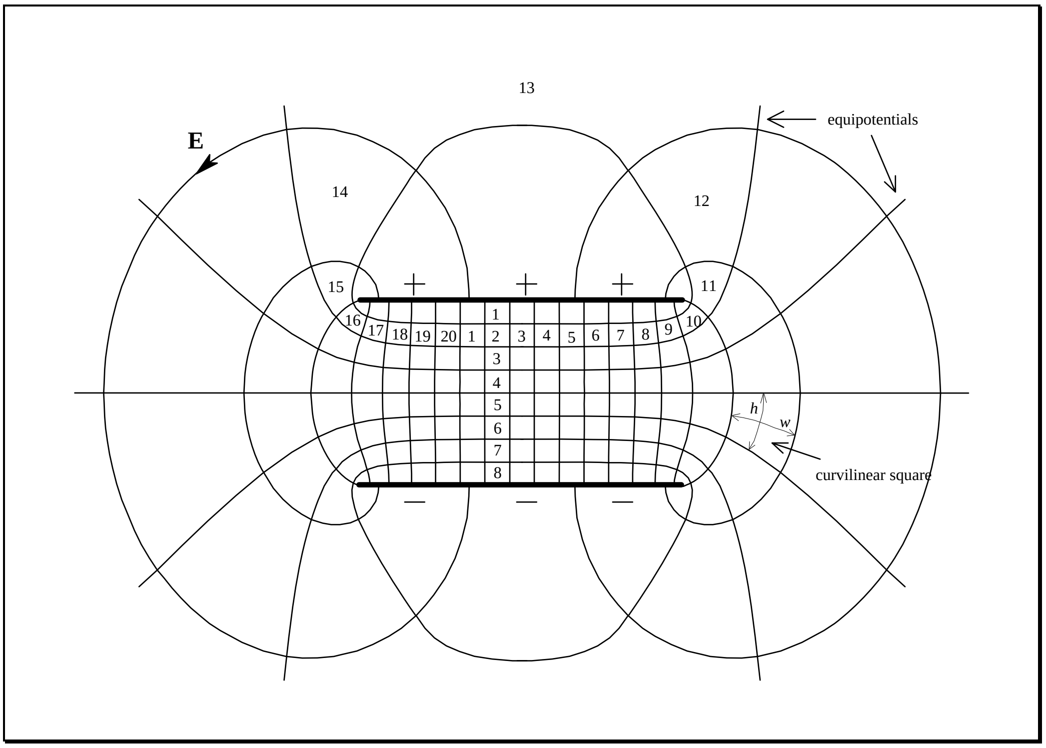


Figure 3B.6

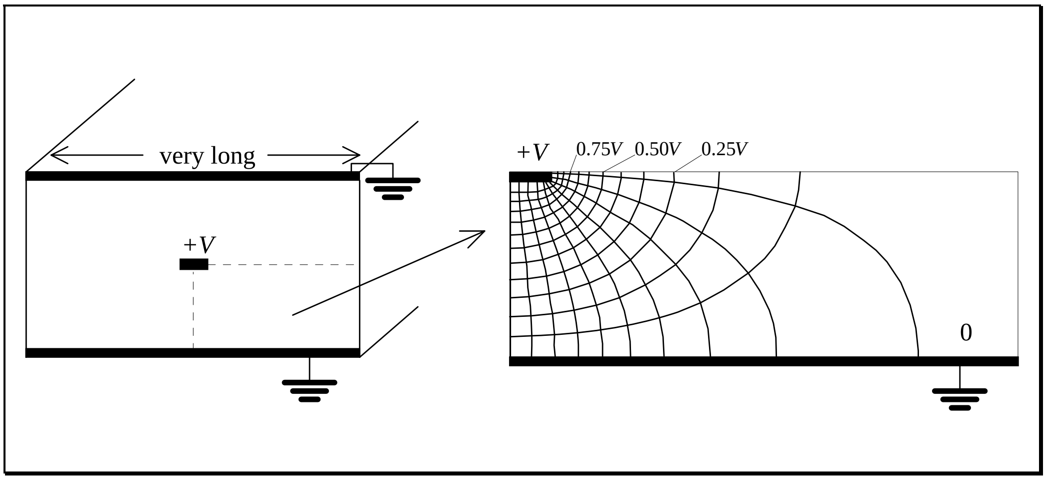
The curvilinear squares in *series* are numbered going from the top plate to the bottom plate. The curvilinear squares in *parallel* are also numbered as we follow one set of equipotential boundaries around the top plate. Even though the fringing field extends to infinity, and some cells are outside the figure, the field map informs us that there are 20 cells in parallel and 8 in series. We therefore have:

$$\frac{C}{l} = \varepsilon \frac{n_p}{n_s} = \varepsilon \frac{20}{8} = \frac{5}{2}\varepsilon \ \text{Fm}^{-1}$$

Of course, practical capacitors do not have a uniform dielectric surrounding them – they usually have a dielectric with $\varepsilon > 1$ sandwiched between the plates which would reduce the fringing field from the sides and outside of the capacitor. Field plots with varying dielectrics are best left to computers…

Example – Rectangular conductor between two earth planes

Consider a rectangular conductor between two earth plates. Due to the symmetry of the arrangement, only a $1/4$ of the field needs to be plotted:



When field plotting, we exploit symmetry

Figure 3B.7

The capacitance per unit length in this case is:

$$\frac{C}{l} = \varepsilon \frac{n_p}{n_s} = \varepsilon \frac{4 \times 9.5}{12} = \frac{19}{6} \varepsilon \ \text{Fm}^{-1}$$

Example – Cylindrical conductor inside metal duct

Due to the symmetry of the arrangement, only $1/8$ of the field needs to be plotted. The surfaces of the inner conductor and of the duct are assumed to be perfect equipotentials.

A mathematically difficult problem made easy

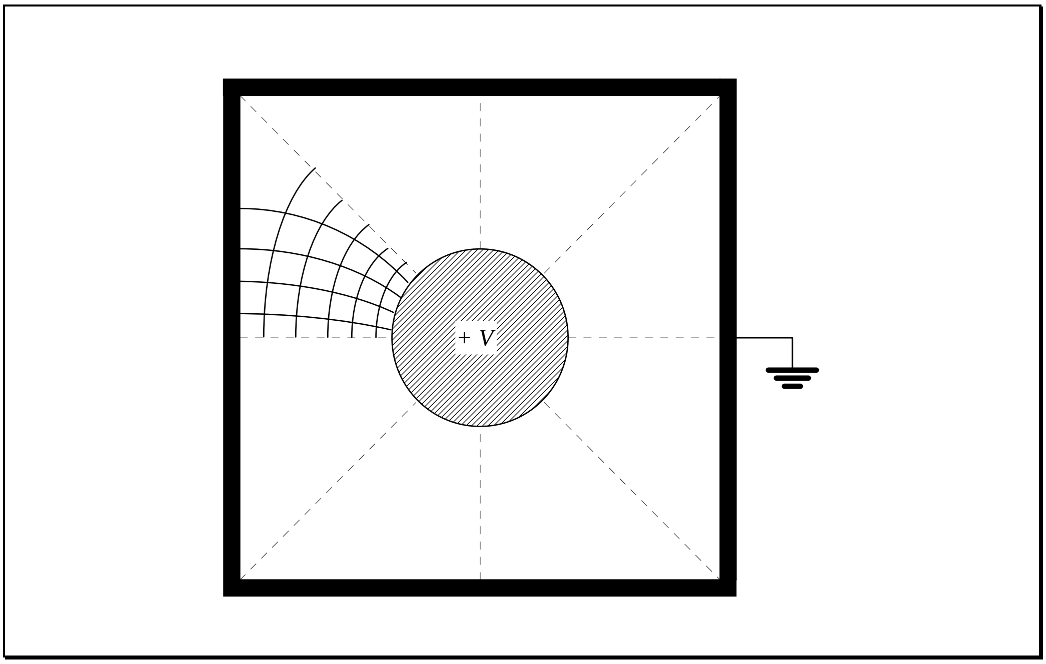


Figure 3B.8

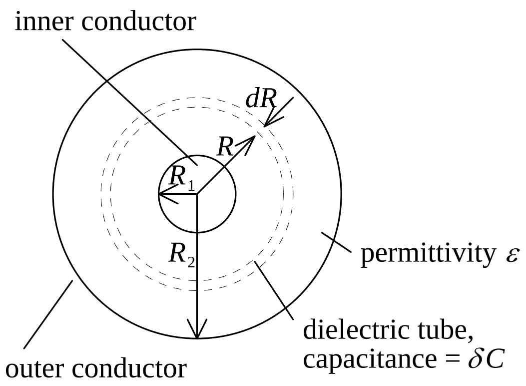
Calculate the capacitance per unit length for the above arrangement.

The Coaxial Cable

A long coaxial cable has a simple symmetry and can be approximated by an infinitely long cable. We have seen it before in the problems. You can derive the formula for capacitance per unit length analytically using the method of curvilinear squares and compare it with that obtained by finding the electric flux density, electric field, voltage and then capacitance per unit length as done previously.

A long co-axial cable is approximated by one of infinite length – so we can plot the field

The method we use is identical to that used to determine the dielectric resistance of a co-axial cable. The dielectric may be assumed to consist of a very large number of concentric tubes, each with a tiny thickness:



A mathematically easy problem is used to verify the method of curvilinear squares

Figure 3B.9

For the dielectric tube shown:

$$\delta C = \frac{\psi}{\delta V} = \frac{\varepsilon E \cdot 2\pi Rl}{\delta V} \approx \frac{\varepsilon 2\pi Rl}{\delta R} \qquad (3B.17)$$

The capacitance of a cylindrical tube with finite thickness

where l = length of the cable.

In the limit, for an infinitesimally thin flux tube, the capacitance is:

The capacitance of a cylindrical tube with infinitesimal thickness

$$dC = \frac{\varepsilon 2\pi Rl}{dR} \tag{3B.18}$$

As all the tubes of flux are concentric, the capacitances *dC* are in series and:

The tubes are added in series to give the total capacitance

$$\frac{1}{C} = \int \frac{1}{dC} = \int_{R_1}^{R_2} \frac{dR}{\varepsilon 2\pi Rl} = \frac{1}{2\pi\varepsilon l} \int_{R_1}^{R_2} \frac{dR}{R} = ? \tag{3B.19}$$

Complete the analysis to determine a formula for C.

The Two Conductor Transmission Line

To calculate the capacitance between two infinitely long conductors, we assume an electrostatic situation – we ignore any current in the conductors and analyse the effect of the charge that has drifted to and remained on the conductor surface. Since we assume a static state of the charge, the surface of the conductor is an equipotential. We then model the surface charge as a line charge at the centre of the conductor:

A surface charge is modelled by a line charge

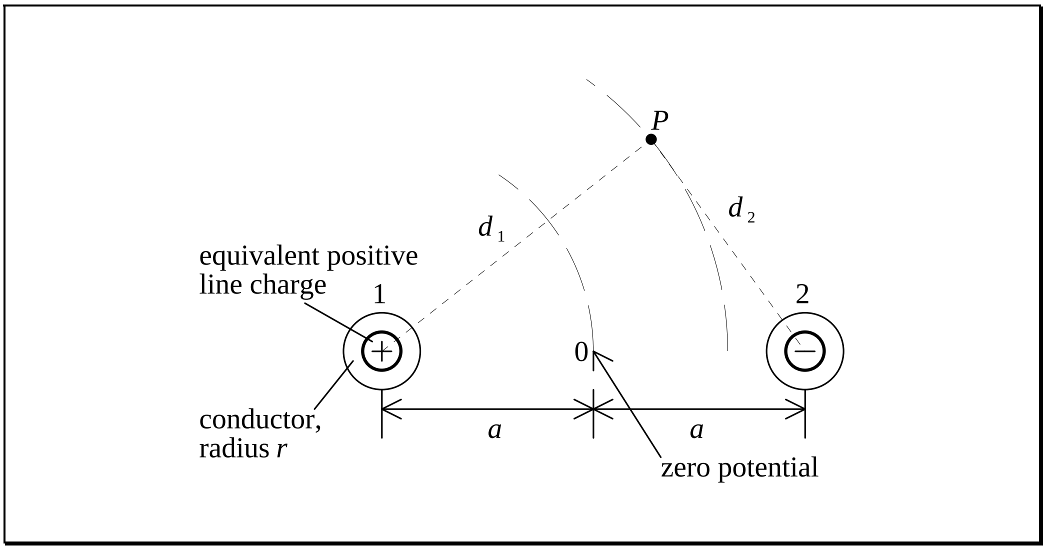


Figure 3B.10

The magnitude of the electric field at radius x due to the positive line charge is:

$$E = \frac{\lambda}{2\pi\varepsilon x} \tag{3B.20}$$

where λ = charge / unit length. The electric potential at point P is:

$$V_P = -\int_0^P \mathbf{E} \cdot d\mathbf{l} = -\int_a^{d_1} \frac{\lambda}{2\pi\varepsilon x} dx = \frac{\lambda}{2\pi\varepsilon} \ln \frac{a}{d_1} \tag{3B.21}$$

The potential at a point due to one conductor

Note that the point of zero potential is arbitrarily taken to be midway between the conductors. (It does not matter where we define “zero” potential, since the only meaningful concept is potential *difference*).

By superposition (assuming a linear medium, such as air), due to both line charges, we get:

The potential at a point due to both conductors

$$V_P = \frac{\lambda}{2\pi\varepsilon} \ln \frac{d_2}{d_1} \tag{3B.22}$$

as the total potential at point P.

The voltage on the surface of the positive conductor (radius r) is similarly given by:

The approximate potential at the surface of a conductor

$$V_1 \approx \frac{\lambda}{2\pi\varepsilon} \ln \frac{2a}{r} \quad (\text{if } 2a >> r) \tag{3B.23}$$

The capacitance per unit length between conductor 1 and the zero potential line is therefore:

The capacitance per unit length between one conductor and zero potential

$$\frac{C_{10}}{l} = \frac{2\pi\varepsilon}{\ln(2a/r)} \tag{3B.24}$$

By symmetry, the capacitance per unit length between conductors 1 and 2 is:

The capacitance per unit length between the two conductors

$$\frac{C_{12}}{l} = \frac{1}{2}\frac{C_{10}}{l} = \frac{\pi\varepsilon}{\ln(2a/r)} \tag{3B.25}$$

(i.e. $C_{10} = C_{20}$ and in series).

The field between conductors of different radii is handled in the same way as the transmission line – an equivalent line charge is located somewhere inside the conductor so that the surface of the conductor is an equipotential:

An equipotential can be used as the surface of a conductor

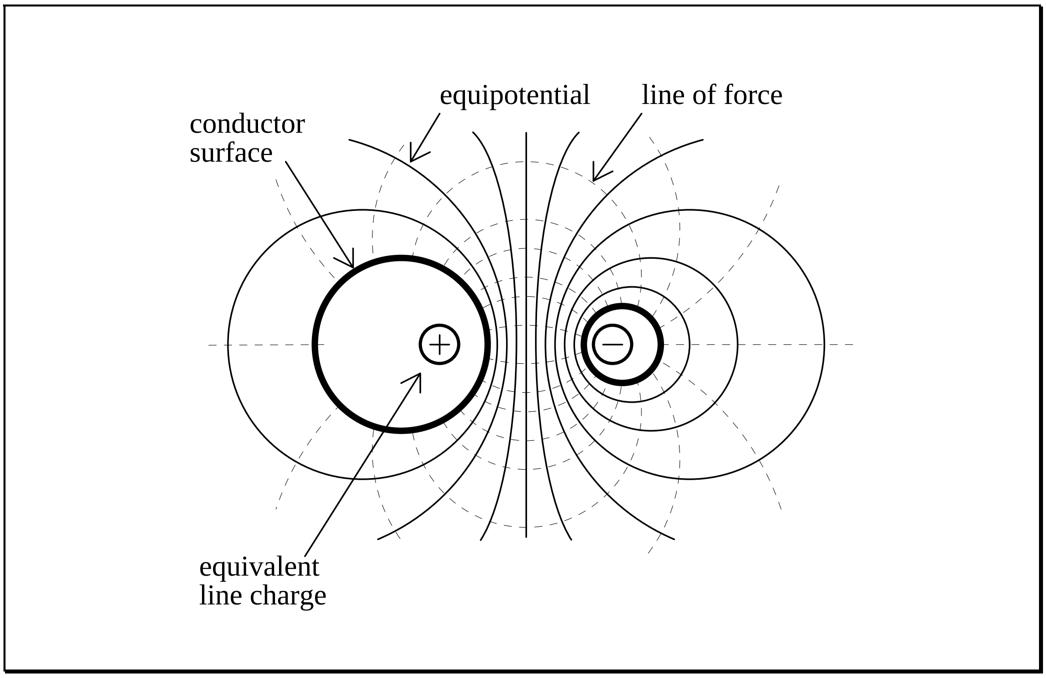


Figure 3B.11

Summary

- A field plot is a plot of equipotentials and lines of force. Two dimensional plots are normally done on paper or a computer.
- Field plots use the concept of a curvilinear square – a shape which has curved sides of roughly equal length.
- Field plots can be used to estimate the capacitance per unit length of irregular shapes and arrangements of conductors.

References

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