9 Coulomb's Law

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Introduction

The study of electrical and magnetic field quantities makes use of physical quantities which possess both magnitude (size) and direction, i.e. directed quantities. You will have met such quantities previously in the form of force, velocity, etc. However, in work involving electric and magnetic fields we must introduce directed quantities in a slightly more formal manner using a system of algebra called Vector Algebra which governs mathematical operations on them. Vector algebra is one of several systems of algebra which you will meet, possibly for the first time, in the study of electrical engineering. The other systems include Boolean Algebra, which governs operations on binary logic variables, and Complex Algebra which is used in the analysis and design of alternating current systems.

Like all systems of algebra, vector algebra is based on a series of postulates (original assumptions) and is developed from these. If these assumptions actually reflect behaviour in the real world then our system of algebra will be of practical use. If not, it is of academic interest only. Vector algebra is of the former kind. It is a mistake to try to relate the rules of one system of algebra directly to another as they will generally be based on different initial postulates. Hence vector algebra has very little in common with our normal numerical algebra and you will find the rules of addition and multiplication completely different in both systems.

In our introduction to vector algebra we study only the rules governing addition, subtraction and multiplication. We will observe these rules with the aid of practical examples.

9.1 Vectors and Scalars

9.1.1 Vectors

A vector is a physical quantity such as force or velocity which has the dual properties of size (magnitude) and direction (line of action) at a given point in space. Both properties may be fixed or be functions of position and/or time. To be completely specified, a vector quantity requires both properties to be stipulated. From time to time we may talk about and perform calculations on the properties separately but ultimately both must be specified.

9.1.2 Scalars

A scalar may be a physical quantity which possesses magnitude only. Examples of common scalar quantities are mass, work, power etc. These quantities may often appear together with vectors in equations relating vector and scalar quantities.

9.2 Vector Algebra

9.2.1 Addition of Vectors

We will use the example of the addition of forces to determine a resultant force. When two (or more) forces act simultaneously on a body the resultant force on the body is obtained from the following procedure.

1. Construct a "polygon of forces" in the plane of the forces where the sides of the polygon represent each force successively in magnitude and direction.

2. The resultant force is given in magnitude and direction by the side which closes the polygon obtained by starting at the 'tail' of the first force and finishing at the 'head' of the last force.

Observe the example below for two forces acting on a body as shown.

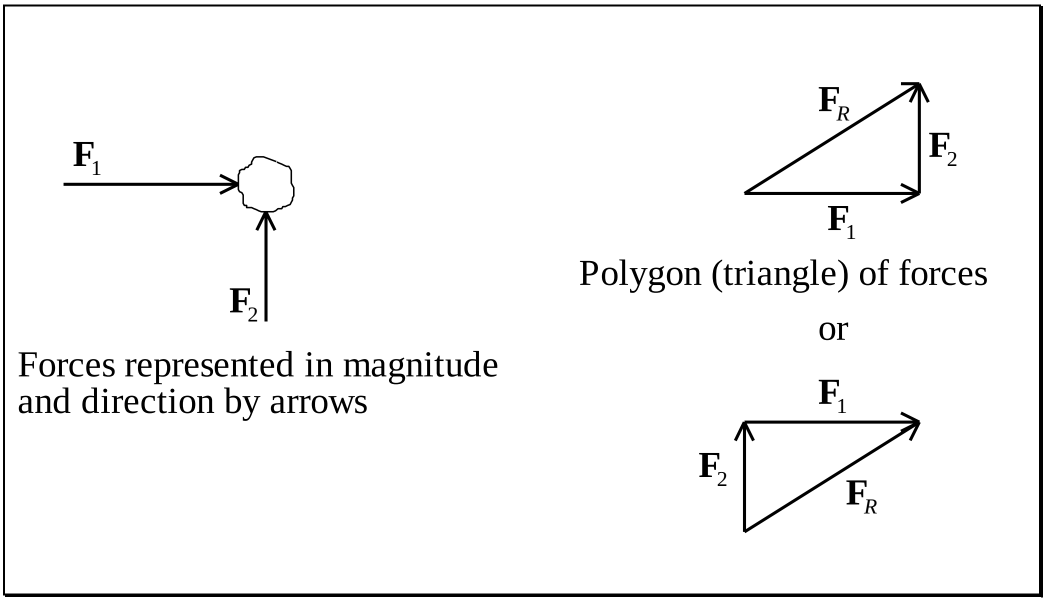

Figure 9.1 – Addition of vectors

Note that $\mathbf{F}_R$ is identical in each case even though the triangle may appear different.

In vector algebra we say:

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= \mathbf{F}_2 + \mathbf{F}_1\end{aligned} \tag{9.1}$$

The symbolism for a vector involves the use of boldface letters. When writing vectors by hand, a curly line is placed underneath the symbol to indicate a vector quantity:

Printed material	Hand-written material
$\mathbf{F}_1$	$\underset{\sim}{F}_1$
boldface	curly line underneath

Figure 9.2 –Vector symbolism

It is obvious that the length of the arrow representing the vector $\mathbf{F}_R$ is less than the straight algebraic sum of the lengths of the arrows representing the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$. So it is obvious also that the symbol '+' in vector algebra must mean something quite different from what it does in normal numerical algebra. This difference must be kept in mind whenever vector quantities are added or subtracted. To obtain the resultant magnitude of a vector addition we must perform a geometrical calculation on the vector quantities involved, taking account of both their magnitudes and lines of action. In the example shown we would be required to use Pythagoras Theorem (if the angle between the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ were a right angle) or else the Cosine Rule. For vectors located in the same plane, resolution into two components mutually at right angles may provide the quickest means of calculating resultants.

9.2.2 Subtraction of Vectors

Subtraction involving vector follows similar rules to addition. For example, we saw previously that $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ and that the resultant had to be determined geometrically. We may reorganise this equation to give:

$$\mathbf{F}_2 = \mathbf{F}_R - \mathbf{F}_1 \tag{9.2}$$

The evaluation of the right hand side of this equation would be performed geometrically:

$$\mathbf{F}_2 = \mathbf{F}_R + \left(-\mathbf{F}_1\right) \tag{9.3}$$

Note that the negation of a vector does not change its magnitude but simply reverses its direction.

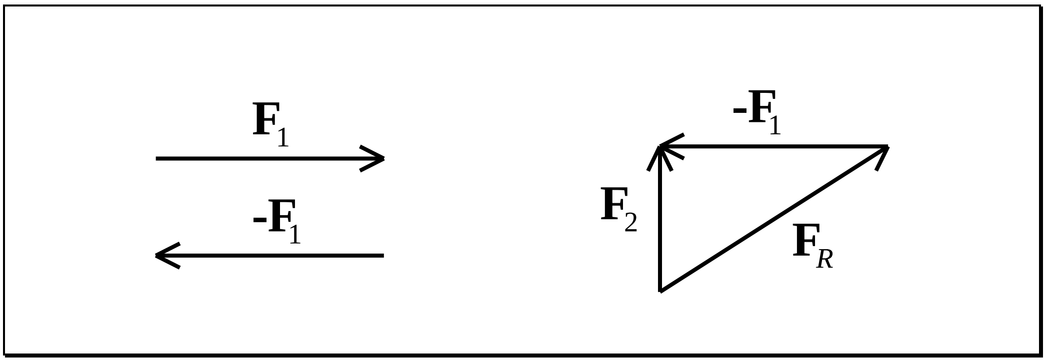

Figure 9.3 – Subtraction of vectors

9.2.3 Vector Multiplication

There are three types of vector multiplication of interest to us:

1. Multiplication of a vector quantity by a scalar quantity.

2. Multiplication of a vector quantity by another vector quantity to give a **scalar** resultant.

3. Multiplication of a vector quantity by another vector quantity to give a **vector** resultant.

Multiplication of a Vector by a Scalar

This process is relatively straightforward. To multiply a vector by a scalar the magnitudes of each are multiplied arithmetically and the line of action of the resultant becomes that of the original vector. Multiplication of a vector by a scalar simply alters the magnitude of the original vector without altering its direction (line of action). See the example given below:

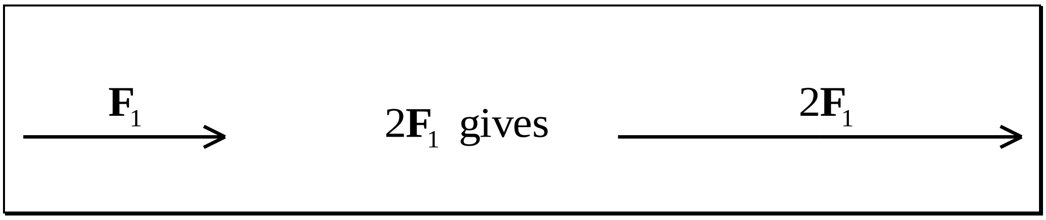

Figure 9.4 – Scalar multiplication of a vector

Scalar Product of Two Vectors

Certain physical quantities are the result of the scalar product of two vectors. One common example of such a physical quantity is mechanical work. When a force (a vector) moves through a distance (a vector) it does mechanical work (a scalar) which is given by:

$$W_{M_{A\to B}} = \int_A^B \mathbf{F}\cdot\mathbf{ds} \tag{9.4}$$

or alternatively:

$$dW_M = \mathbf{F}\cdot\mathbf{ds} \tag{9.5}$$

The multiplication process is symbolised by the dot appearing between the two vectors. For this reason the scalar product is often called the 'dot' product.

You will have realised that the mechanical work equation has been expressed in terms of the differential dW and the differential vector $\mathbf{ds}$. This is an important departure from what you may have experienced previous1y, but it is an essential refinement of the equation:

$$W_M = \mathbf{F}\cdot\mathbf{s} \tag{9.6}$$

since this equation is only correct if the magnitude and direction of $\mathbf{F}$ are constant over the straight directed path $\mathbf{s}$. In many cases we will examine, $\mathbf{F}$ is neither constant in magnitude or direction, neither is the path moved by $\mathbf{F}$ a straight one.

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta} \tag{9.7}$$

In this equation, the magnitude of a vector is denoted by the absolute value sign (the quantity within | |). For vectors, it is usual to let their magnitude be

symbolised by the same letter, but using italics instead of boldface. Thus, the magnitude of a vector $\mathbf{x}$ can be represented as:

$$|\mathbf{x}| = x \tag{9.8}$$

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can then be written as:

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta \tag{9.9}$$

The angle θ is the angle described in moving from the direction of vector $\mathbf{a}$ to that of vector $\mathbf{b}$, as shown below:

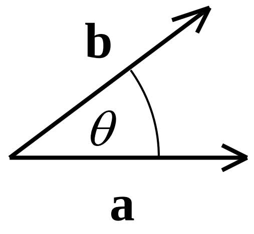

Figure 9.5 – Scalar product

Obviously if $\theta = 90°$:

$$\mathbf{a} \cdot \mathbf{b} = ab\cos 90° = 0 \tag{9.10}$$

Also, since $\cos(-\theta) = \cos\theta$, we have:

$$\boxed{\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}} \tag{9.11}$$

Vector Product of Two Vectors

The vector product of two vectors is not as common as the scalar product at this stage of knowledge of Physics. But there is one physical quantity which is reasonably common and which is the result of a vector product of two vectors. This quantity is mechanical torque, $\mathbf{T}$:

$$\mathbf{T} = \mathbf{s} \times \mathbf{F} \tag{9.12}$$

In this case the multiplication process is symbolised by a cross appearing between the vectors $\mathbf{s}$ and $\mathbf{F}$ and as before the vector product is commonly known as the 'cross' product.

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\boxed{\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}} \tag{9.13}$$

where θ is the angle described in moving from the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a vector of unit magnitude whose line of action is the direction that a right-hand screw would move in rotating through the angle θ. In the figure below, vectors $\mathbf{a}$ and $\mathbf{b}$ are in the horizontal plane, vector $\hat{\mathbf{n}}$ is at $90°$ to both vectors $\mathbf{a}$ and $\mathbf{b}$.

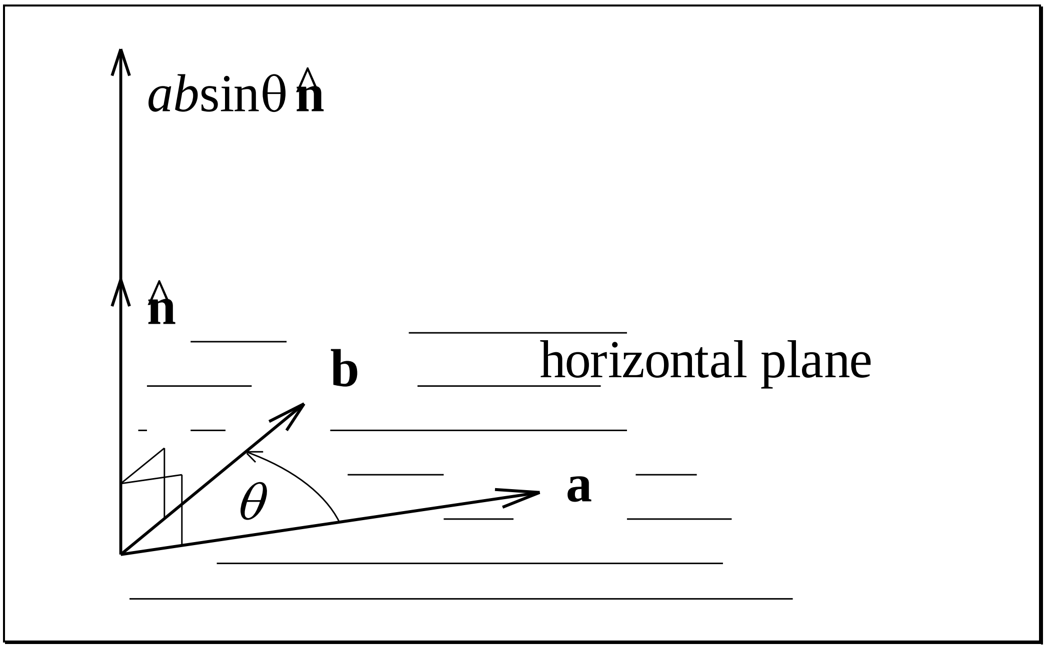

Figure 9.6 – Vector product

Obviously if $\theta = 0°$:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin 0° \, \hat{\mathbf{n}} = 0\hat{\mathbf{n}} \tag{9.14}$$

which is a vector of zero magnitude.

The order in which vectors appear in the cross product is important since $\sin(-\theta) = -\sin\theta$. That is:

$$\boxed{\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}} \tag{9.15}$$

You may wish to show this is true by yourself.

9.2.4 The Unit Vector

You will have observed in the discussion above that the vector $\hat{\mathbf{n}}$ was introduced with unit magnitude and a specified direction. A vector such as this is called a 'unit vector'. It is distinguishable by the circumflex (^) appearing above the vector. The unit vector is a very convenient method of introducing the "direction" information for a vector quantity. The magnitude of a vector is of course scalar (magnitude has no direction) so that if the unit vector (which contains all the direction information) is multiplied by a scalar magnitude then the resulting vector quantity will be a vector with magnitude equal to the scalar magnitude and direction given by that of the unit vector. That is:

$$\mathrm{n}\hat{\mathbf{n}} = \mathbf{n} \tag{9.16}$$

or alternatively:

$$\boxed{\hat{\mathbf{n}} = \frac{\mathbf{n}}{\mathrm{n}}} \tag{9.17}$$

The concept of the unit vector will be used often in developing electric and magnetic field vector relationships later.

9.3 Coulomb's Law

In 1767 Priestley deduced a $1/r^2$ distance dependence for separated electrostatic charges which was similar to Newton's Law of Gravitation. The literature describes European experiments with suspended point charges enclosed by hollow charged conductor spheres. No force was experienced on such an enclosed charge irrespective of its position within the sphere:

Priestley's electrostatic experiment

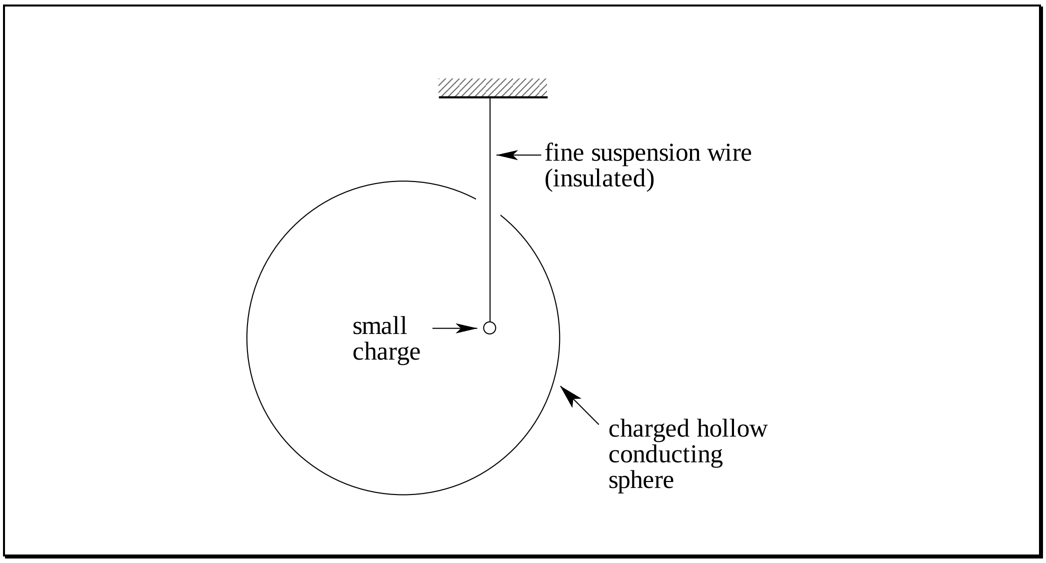

Figure 9.7 – The experimental arrangement that led Priestley to conclude an inverse square law (the charge experienced no motion)

Newton had already shown that there would be no force experienced inside a hollow Earth, and this fact led Priestley to conclude that:

$$F \propto q/r^2 \tag{9.18}$$

where q is the magnitude of the source charge, and r the separation of this charge from the suspended test charge.

This $1/r^2$ electrostatic law was verified by direct experiments of Coulomb in 1785, and in his reports on electricity and magnetism he stated his law:

$$\mathbf{F}_{21} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}_{21} \quad \mathrm{N} \tag{9.19}$$

Coulomb's Law

The quantities expressed in this law are shown below for two positive charges:

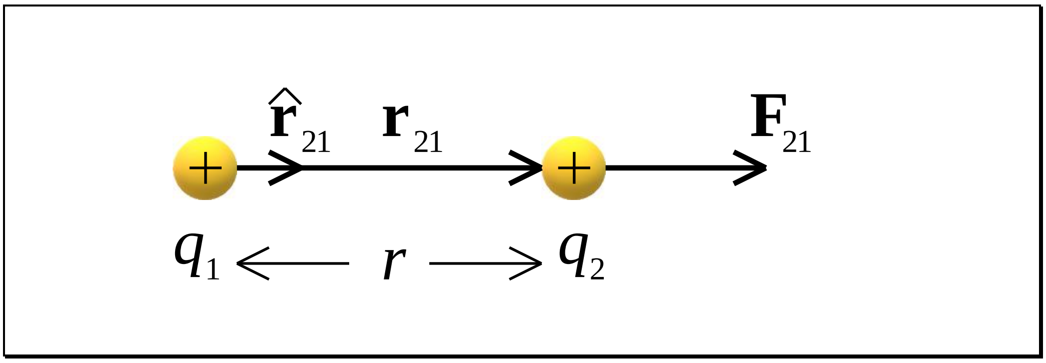

Figure 9.8 – Illustration of terms involved in Coulomb's Law

The subscript ‘21’ on the force means this is the force on charge q_2 due to charge q_1. The radial vector $\mathbf{r}_{21}$ points from q_1 to q_2, and can be more formally expressed as the difference between two “position” vectors that point from an arbitrary origin to each charge in 3D space:

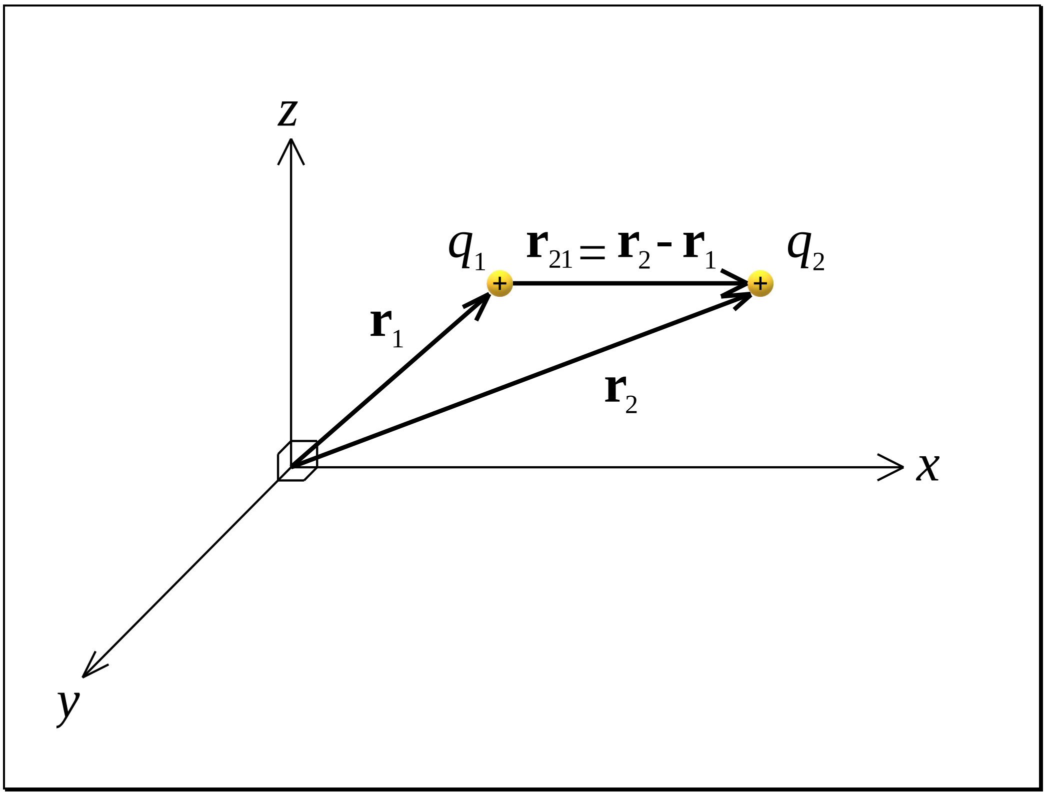

Figure 9.9 – Illustration of coordinate system in Coulomb’s Law

Therefore, the unit vector $\hat{\mathbf{r}}_{21}$ points in a direction that goes from a positive q_1 to positive q_2 (or from a negative q_1 to negative q_2), i.e. the unit vector aligns with the $\mathbf{F}_{21}$ direction of repulsion. Attractive forces will therefore be negative, and are taken care of by the negative sign of either q_1 or q_2.

In the SI system of units, Coulomb’s Law contains a “universal constant” called the *permittivity of free space* which is given by:

$$\boxed{\varepsilon_0 = 8.85419 \times 10^{-12} \text{ Fm}^{-1}} \tag{9.20}$$

9.4 Electric Field Intensity

In Figure 9.8 the charge q_2 will feel a force even though nothing is touching it! We know that it is caused by q_1. We can now imagine a "field of influence" radiating out from charge q_1 into all of space (3 dimensions). As far as q_2 is concerned, it finds itself immersed in a region of space where a force is felt (think of the gravity analogy: the Earth is attracted to the Sun, across a seemingly empty space). We can then imagine that something permeates the space even before we place our charge q_2 in it. We call this a *field*. When we place the charge q_2 in the field, we see a reaction – in this case a force.

Concept of a "field"

With this thinking, it appears that a field exists due solely to q_1. We call this field the *electric field*, and for a point charge q_1 located at the origin the electric field is defined as:

$$\boxed{\mathbf{E}_1 = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad \text{Vm}^{-1}} \tag{9.21}$$

Electric field of a point charge

where $\mathbf{r}$ is a radial vector to a "test" point in 3D space where we want to find the eclectic field, and the subscript on the **E** corresponds to the subscript used on the charge that produces the field (in this case, "1").

9.4.1 Lines of Force

It is useful to have a kind of "map" that gives the direction and indicates the strength of the field at various places. The great English physicist Michael Faraday (1791–1867) proposed an idea that provides such a "map" - the idea of electric field lines. Since the electric field is the electric force per unit charge, the electric field lines are sometimes called *lines of force*.

Electric lines of force

For an isolated point charge the electric field must exist all around it, and be spherically symmetric – i.e. there is no way to orient a point, so the effect it produces must be spherical in nature. Now take a very small positive test charge and place it in the field near the point charge. We will constrain the test

charge to move infinitesimally slowly away from the point charge. The path the test charge traces out follows a line of force.

For an isolated positive point charge, the lines of force radiate from the charge in all directions:

Lines of force give a "picture" of the field

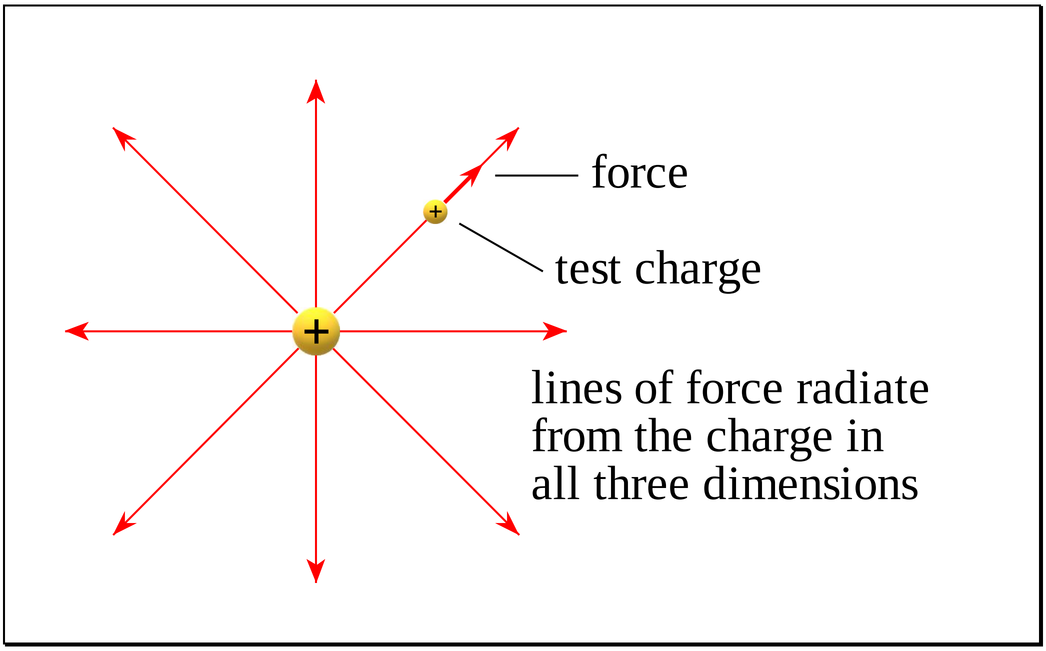

Figure 9.10 – Lines of force illustrating the electric field of a point charge

In general, electric field lines are always directed away from positive charges and toward negative charges. The lines of force drawn in this manner create a picture of the electric field. The direction of the electric field at any point is given by the direction of the force on the positive test charge.

Using these ideas, we can calculate the force on a charge using the concept of an electric field rather than Coulomb's law:

$$\boxed{\mathbf{F}_{21} = q_2 \mathbf{E}_1} \tag{9.22}$$

The picture of field lines in Figure 9.9 is very simple as only one charge is present. If we now consider the field of an *electric dipole*, which consists of two separated point charges that have the same magnitude but opposite signs, we get lines of force as shown below:

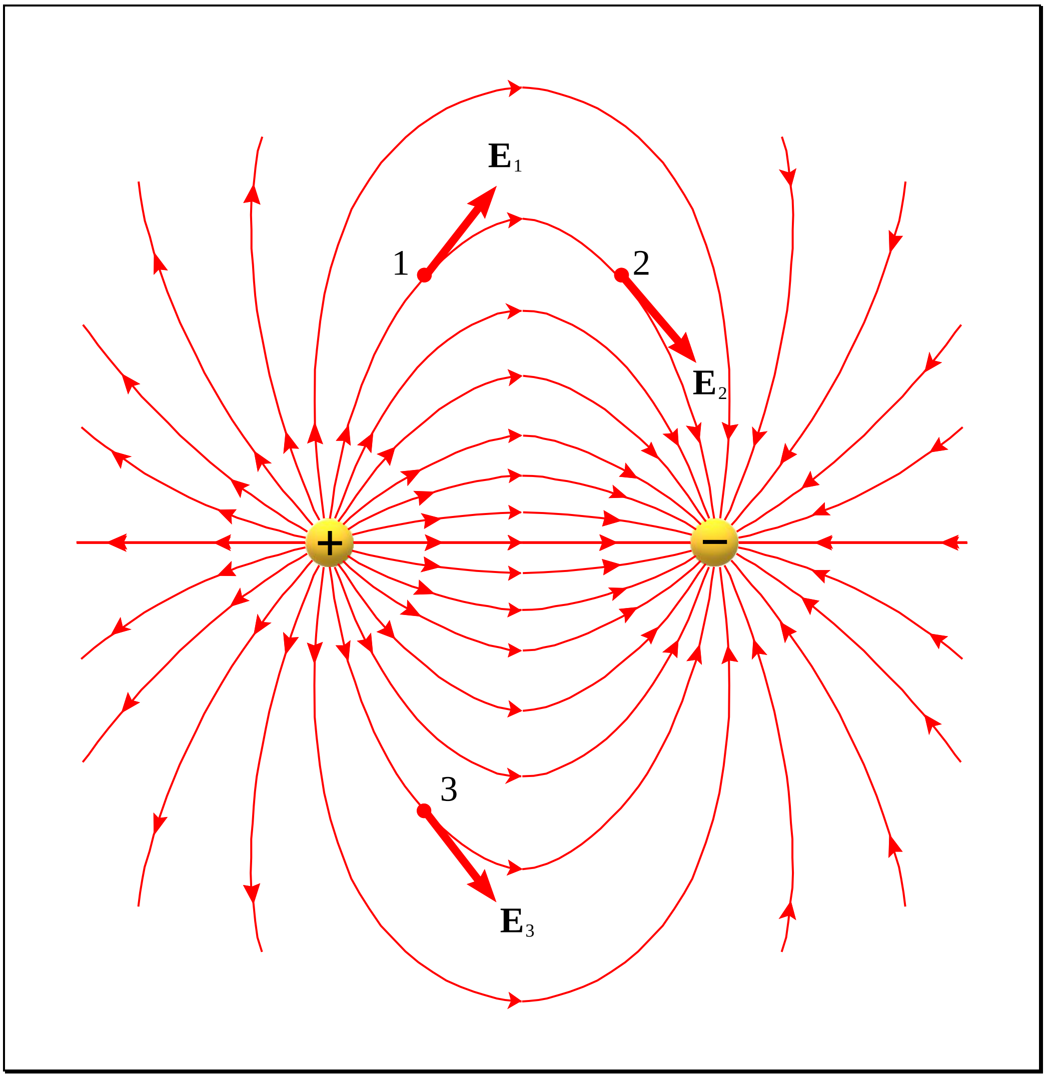

Figure 9.11 – Electric field lines of an electric dipole

As can be seen, for curved field lines the direction of the electric field vector **E** at any point is given by the *tangent* to the line of force at that point (see points 1, 2 and 3 in the figure).

The pattern of electric field lines also provides information about the magnitude or strength of the field. Notice in Figure 9.10 and Figure 9.11 that near the charges, where the electric field is stronger, the lines are closer together. At distances far from the charges, where the electric field is weaker, the lines are more spread out. It is true in general that the electric field is stronger in regions where the field lines are closer together. In fact, no matter how many charges are present, the number of lines per unit area passing perpendicularly through a surface is proportional to the magnitude of the electric field.

Notice in Figure 9.11 that any given field line starts on the positive charge and ends on the negative charge. In general, electric field lines always begin on a positive charge and end on a negative charge and do not start or stop in midspace, except where $\mathbf{E} = \mathbf{0}$. Furthermore, the number of lines leaving a positive charge or entering a negative charge is proportional to the magnitude of the charge.

Since there is only one direction of electric field intensity **E** at any point, lines of force cannot intersect, except where $\mathbf{E} = \mathbf{0}$. For isolated charge, assume there is an equal and opposite charge located at infinity (conservation of charge).

An example of a field map where there is a point where $\mathbf{E} = \mathbf{0}$ is shown below. In this case there are two equal positive charges (the field map would look the same with two equal negative charges, but with the direction of the field lines reversed).

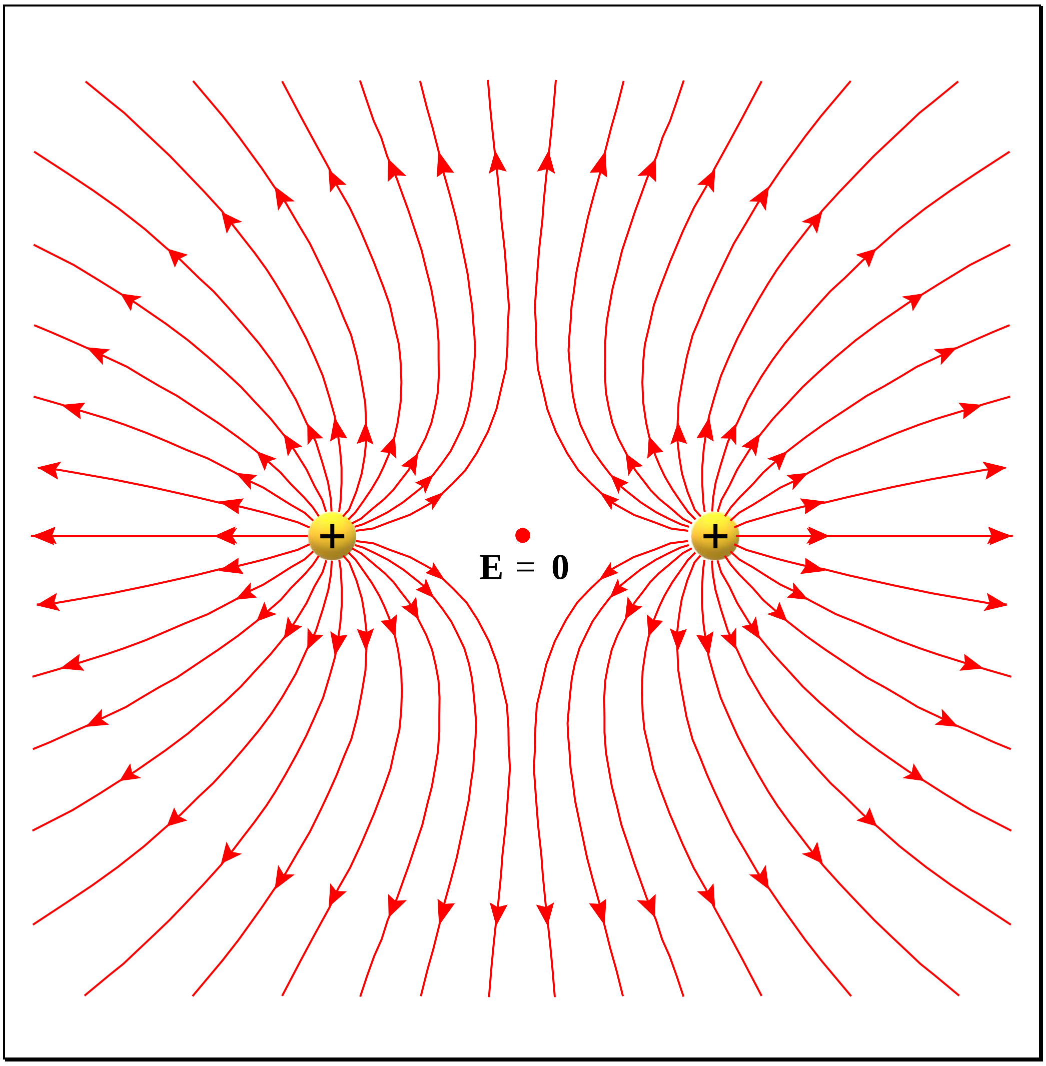

Figure 9.12 – Electric field lines for two identical positive point charges

The electric field is zero at the midway point connecting the two charges. This should coincide with your intuition that midway between the charges a test charge would experience equal force magnitudes due to both charges, but in opposite directions. The field $\mathbf{E}$ is finite at all other points in this field map.

9.4.2 Superposition

Superposition applied to components of a force

In mechanics we often split up a total force on an object into a number of components. Conversely, we can add up a number of components to get the total force. We can do the same with the electrostatic force. Consider the following arrangement:

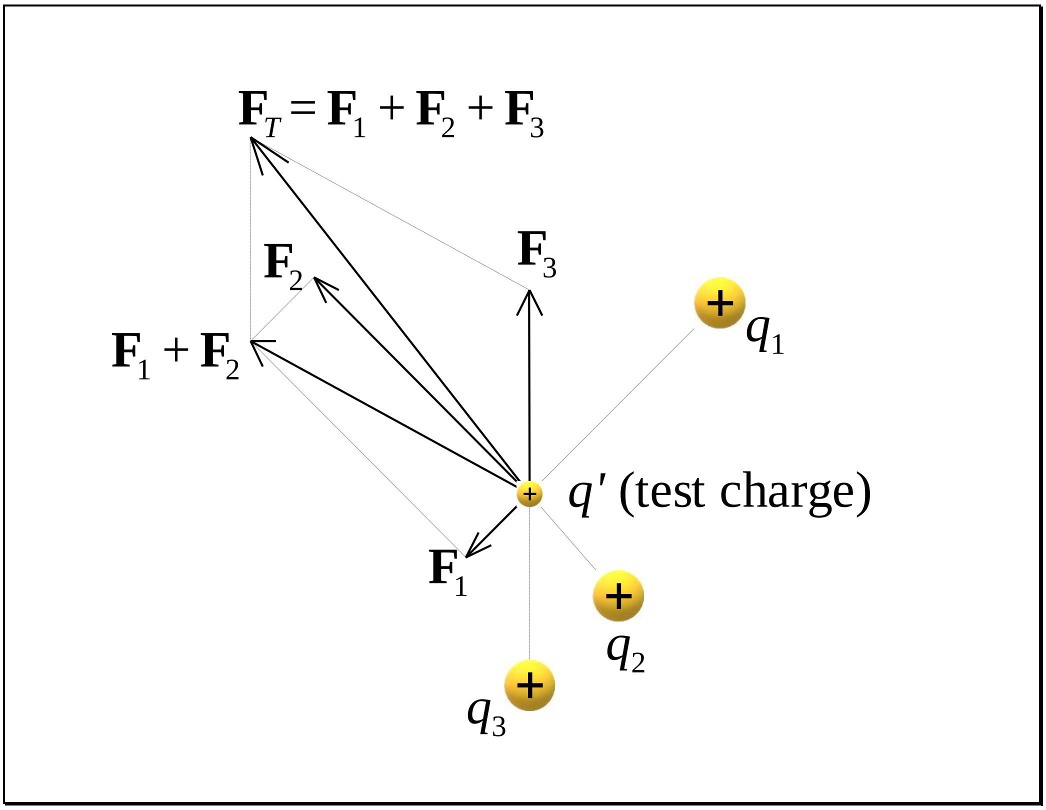

Figure 9.13 – Superposition of electric forces

The resultant force on the test charge is given by:

Using superposition to calculate the field due to more than one charge

$$\mathbf{F}_T = \sum_{i=1}^{3} \mathbf{F}_i = q' \sum_{i=1}^{3} \mathbf{E}_i = q' \mathbf{E}_T \tag{9.23}$$

This shows that the electric field obeys the principle of superposition, and the total electric field is just the sum of the fields due to individual charges:

$$\boxed{\mathbf{E}_T = \sum_{i=1}^{n} \mathbf{E}_i} \tag{9.24}$$

9.5 Summary

- Addition and subtraction of vector quantities is performed geometrically, since direction needs to be taken into account.

- There are three types of vector multiplication:

  1. Multiplication by a scalar: $k \times \mathbf{a} = k\mathbf{a}$.

  2. The 'dot' product: $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$.

  3. The 'cross' product: $\mathbf{a} \times \mathbf{b} = ab\sin\theta\,\hat{\mathbf{n}}$.

- A *unit vector* has magnitude 1 and conveys direction information only:

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\mathrm{n}}$$

- The electrostatic force between two infinitesimally small electric charges is given by Coulomb's Law:

$$\mathbf{F}_{21} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}_{21} \quad \mathrm{N}.$$

- The *permittivity of free space,* ε_0, is a universal electric constant.

- An infinitesimally small electric charge produces an electric field given by:

$$\mathbf{E}_1 = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad \mathrm{Vm}^{-1}.$$

- Electric field lines, or lines of force, are used to make a "map" of the electric field. The density of the field lines is proportional to the electric field magnitude, and the tangent to the field lines is conveys the direction of the electric field. All field lines begin with positive charges, and end with negative charges. Lines of force do not cross.

9.6 References

Beard, G.: *Electrostatics and Magnetostatics*, NSWIT, 1985.

Bryce, P.: *Field Theory 2 Notes*, NSWIT, 1988.

Exercises

1.

Calculate the distance of separation between two electrons (in vacuum) for which the electric force between them is equal to the gravitational force on one of them at the Earth's surface.

2.

Point charges of 2×10^{-19} C are situated at each of three corners of a square whose sides are 0.2 m.

(a) What would be the magnitude and direction of the resultant force on a point charge of -1×10^{-9} C if it were placed at the centre of the square, and what is the field intensity at this point?

(b) Find the field intensity at the vacant corner of the square, and the force on -1×10^{-9} C placed at this point.

3.

Calculate the intensity of an electric field at a point (P), 30 mm from a positive charge of 8×10^{-6} C when this point (P) is also 60 mm away from a negative charge of -8×10^{-6} C. The two charges are separated by a distance of 90 mm.

4.

A small sphere of mass 2×10^{-3} kg and charge 6×10^{-9} C hangs by a thread in a uniform horizontal field **E**. What is the magnitude of **E** if the thread assumes an angle of 30° to the vertical?