10 Gauss' Law

Contents

10.2

Introduction

The study of fields is concerned with predicting interactions between things that are not apparently touching. There are a wide variety of events that cannot be explained by mechanical contact forces, and the problem of adequately visualising such 'actions at a distance' has been with us for several hundred years.

We will presently find that the model we use to explain the electrostatic force will naturally extend to a variety of other uses, particularly in electrodynamics, (electro) magnetic machines, power transmission, electrical communications (especially fiber optics, broadcasting, and radio frequency (RF) communications such as WiFi/Bluetooth etc.), high-frequency interference in circuits and high-voltage corona.

In addition, the model will provide valuable insight into familiar equations used in electrical engineering that are not normally considered as field-related. We can answer some basic questions about electrical behaviour that are normally 'glossed over' or ignored. For example, how is it that a resistor can be influenced (getting hot) by an AC signal generator some distance away? After all, there is no net transport of electrons from one to the other. What is really happening when we connect “circuit elements” such as batteries and resistors together?

In summary, we need to develop a consistent model for events where no mechanical explanation is apparent. The model should give the right answers in all macroscopic interactions (or at least those obeying inverse-square laws like Coulomb’s Law) but also should alleviate our psychological hang-ups regarding action-at-a-distance. The model we use can be vastly simpler than the particle model used in quantum electrodynamics, since we will never attempt to explain sub-atomic effects.

10.1 Models for Action-at-a-Distance

Since no mechanical contact is apparent in the forces between charges, a multitude of models have been proposed in order to visualise the mechanisms of “action-at-a-distance”. The main hypotheses that have been in vogue from time to time are:

Stress-Through-the-Medium

Stress waves may be transmitted through an intermediary medium (the *ether*) by the source particle and picked up by the other particle as a force.

Particle Emission

The interaction of bodies at a distance may be via the exchange of particles. Each body may emit particles which by "collision" transfer a force to the other.

Fluid Emission

Interaction at a distance may be via a fluid, independent of any medium. Each body may emit the fluid, in order to transmit forces to neighbouring bodies.

It turns out that to explain all the macroscopic properties observed in the electromagnetic world, the fluid model is the most fruitful and has stood the test of time (quantum electrodynamics is better but too complicated!).

10.2 The Fluid Emission Model

The idea of an all pervading 'ether' must have seemed quite plausible before Michelson and Morley[1], despite being far-fetched when viewed by hindsight. The notion of fluid emission seems just as absurd, in an *experimental* sense. However, to explain all the macroscopic properties observed in the electromagnetic world, the fluid needs just three simple properties (the combination of these properties *can* be observed):

Postulate 1

All force-producing points (or particles) emit fluid ISOTROPICALLY.	(10.1)

Isotropic means uniform in all directions. The points may be the things we commonly term 'charges'. The 'fluid' differs from usual fluids according to this postulate, because we would expect most other fluids to show anisotropy when the source moves. Accordingly, we normally call this special fluid a special name, *flux*, symbolised by Ψ . It is a scalar, just like the volume (and flow rate) of any fluid.

Postulate 2

The density of flux Ψ (per unit cross-sectional area) found at any point in space is proportional to the force transmitted to that point.	(10.2)

That is, Ψ is a scalar, indicating the volume of 'fluid' emitted, but the density of Ψ must be a vector.

The density **D**, in units of flux per unit area, can be used to give us the magnitude and direction of the force potentially experienced (by another charge) at each point in space.

Postulate 3

The total flux emitted from a point source is not changed if the source is at rest or in motion.	(10.3)

[1] See https://en.wikipedia.org/wiki/Michelson–Morley_experiment

We will find these three simple notions adequate to explain all our (macroscopic) Electrical Engineering Laws, and many more phenomena in engineering. 'Flux' has never been observed as a physical fluid (apart from its three postulated behaviours above) and no doubt never will be.

The model of flux emission satisfies a key property of a good model; it is exceedingly simple. For this reason alone, it has stood the test of time as a very useful mathematical model in those disciplines where a fundamental inverse square-law has been observed.

Finally, we finish this outline of a model for action-at-a-distance by a definition. Every particle, or point source, does not produce the same effect on its neighbours. There must be a quantity defined to express this variable influence:

Definition: The 'strength' of a source of flux is the total flux emitted from the source. The magnitude of this 'source strength' is called the *charge*.

The total Ψ being emitted from a source	=	Charge q at the source

(10.4)

Total flux = charge at the source

Since Ψ is a scalar, so is charge q. In SI units q is in Coulombs. Accordingly, from this definition, Ψ is also in Coulombs.

The 'everyday' quantity of charge has never been observed. The presence of flux Ψ can at least be seen by its influence (transmitted forces), and can therefore be indirectly observed. If you are happy with 'charge', then is it not reasonable to expect that we henceforth take for granted that 'flux' exists?

There are three basic equations used in electrostatics. They can be readily seen as a consequence of the properties of flux (as listed above). Gauss' Law is simply an expression of the meaning of charge. Coulomb's Law must follow for a flux emission that is isotropic. The final equation, Poisson's equation, is a statement that follows from Coulomb's Law and a minor facility with *vector calculus*. Firstly, we consider Gauss' Law, then we will *derive* Coulomb’s Law (which was supposedly a fundamental experimental law). We will delay the consideration of Poisson’s equation to future study.

10.3 Area Vectors

Area vector defined

An *area* vector has the job of specifying the size and direction of an area. Direction of an area? Yes, by convention, the *direction* of an area is defined to be the direction *perpendicular* to the plane of the area. Of course, this implies that the area is flat, but it also applies to curved surfaces that are infinitesimally small, where the vector points in the direction of the *normal*. For example, the area vector for a rectangle would look like:

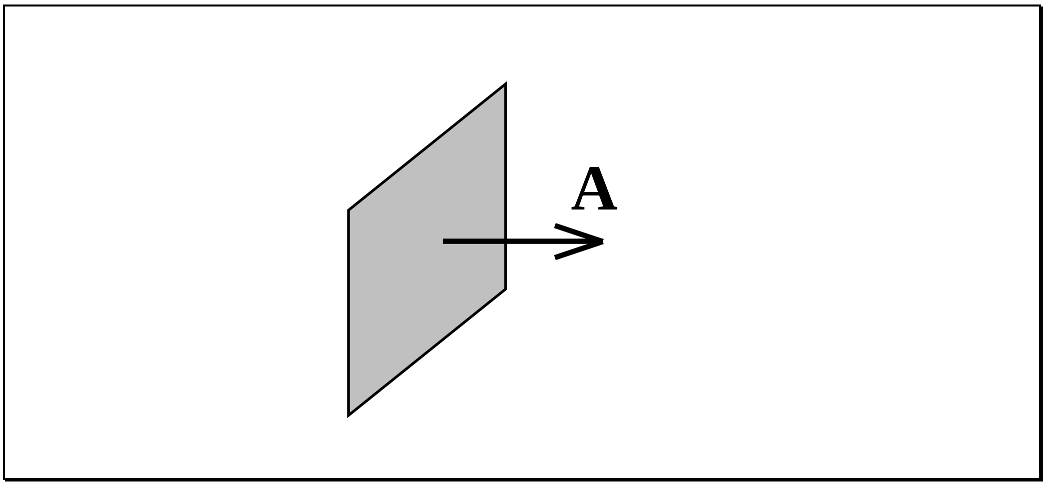

Figure 10.1 – An area vector for a rectangle

where the *magnitude* of **A** would equal the area of the rectangle. An infinitesimally small area, such as part of a sphere would be represented by a radially directed vector **dA**:

Differential area vector

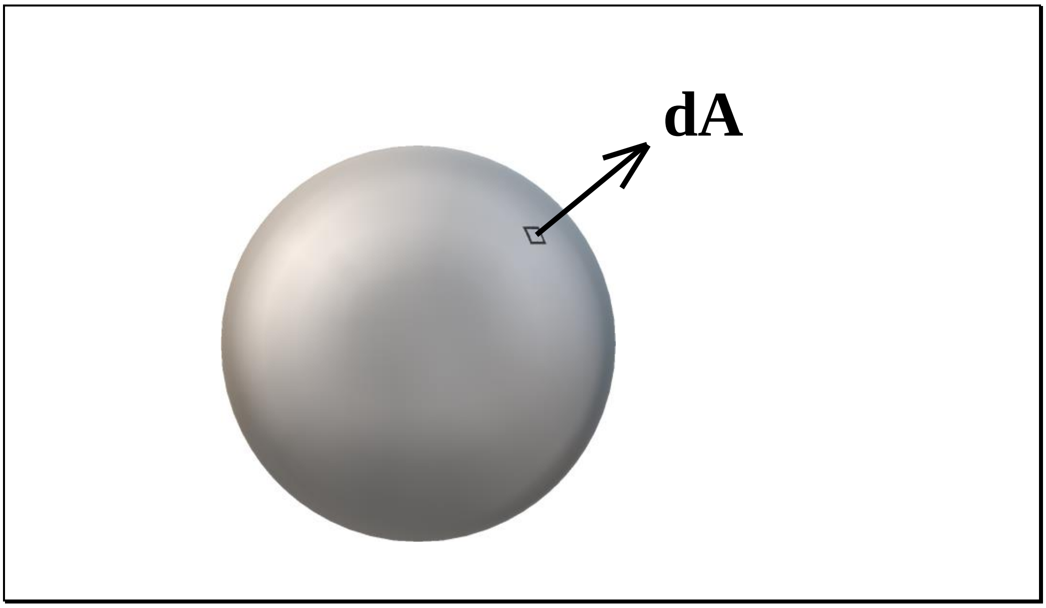

Figure 10.2 – A differential area vector on a sphere

10.4 Gauss' Law

Consider a situation where we have a source of flux (a 'charge', or set of charges) in some volume. We wish to know how much flux is being emitted. We may not be able to see the flux coming out, but we can detect it by its property as defined by Postulate 2. We can measure its influence through the force that it transmits.

So we put a force-measuring instrument, say a 'test' charge attached to a spring balance that responds to the repulsive force, at points just outside the volume. Then, at each point:

$$\text{The measured force } \propto \mathbf{D} \tag{10.5}$$

Given that our measuring spring balance subtends a small area dA, a small amount of fluid will be intercepted by the area. The fluid detected will increase as the area of the instrument's face increases, but will also change with the orientation of the face; see the figure below:

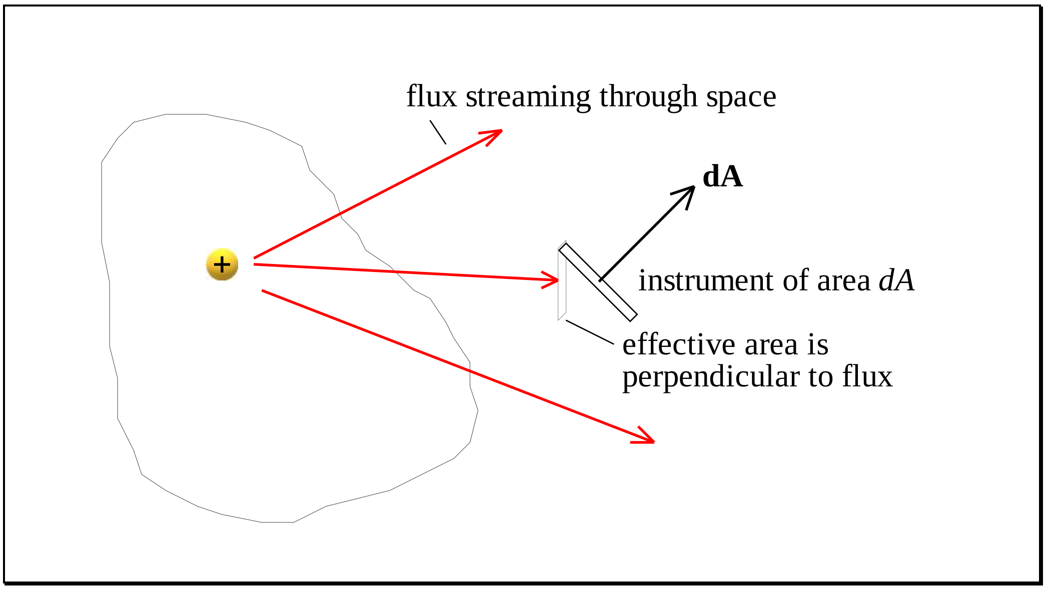

Figure 10.3 – Measuring force due to a flux

The maximum amount of flux will be detected when the face is a plane directly across the flow; that is when $\mathbf{dA}$ is parallel to $\mathbf{D}$. In general, only the component of $\mathbf{dA}$ that is parallel to $\mathbf{D}$ will be effective in intersecting flux lines.

Thus the instrument accounts for a flux of:

$$d\Psi = \mathbf{D} \cdot \mathbf{dA} \tag{10.6}$$

To detect all the flux emitted from the volume, we need to move the detector around the periphery of the volume, to gather the effect of the other flow lines. We must put the detector at points close enough to one another so as not to miss any of the flow, and the positions must extend around the full periphery, without any gaps. Then the sum of all the measured fluxes, over the completely enclosing surface periphery is:

$$\oint \mathbf{D} \cdot \mathbf{dA} = \text{total } \ \Psi \tag{10.7}$$

The circle on the integral means "over a closed surface". By definition (10.4), this total flux emitted is equal to the total source strength within the volume:

Gauss' Law

$$\boxed{\oint \mathbf{D} \cdot \mathbf{dA} = q_{\text{enclosed}}} \tag{10.8}$$

This is Gauss' Law.

Note that if negative charge exists in the volume, there will be negative source strength at such points. That is, the flux emission will be a negative flow rate, and the points then represent 'sinks' of flux. Flux is absorbed in such cases. It could be, then, that fluid may be flowing into the volume through some parts of the enclosing surface, and the detecting spring balance will in fact be stretched rather than compressed. Influences from sinks of fluid will then appear in the integration sum of Gauss' Law as negative terms. The full summation thus gives a nett outward flux.

$$\boxed{\begin{array}{c}\text{Nett outward flux}\\ \text{from a volume}\end{array} = \oint \mathbf{D} \cdot \mathbf{dA} = \begin{array}{c}\text{Nett source}\\ \text{strength in the}\\ \text{volume}\end{array}} \tag{10.9}$$

This notion, called Gauss' Law by Electrical Engineers, is true in many other fields. For other cases, read **D** as the flux density of interest.

10.4.1 The Flux Density Around an Isolated Point Charge

Gauss' Law is, in a great many cases, almost impossible to solve analytically, but there are a handful of useful analytical expressions that exploit the geometry and symmetry of the way charge is distributed in 3D space. The simplest case is that of an isolated point charge. We will set a positive point charge at the origin of our coordinate system and enclose it with a spherical *Gaussian surface*, over which we will apply Gauss' Law. Note that Gauss' Law will work with any enclosing surface (such as a cube), but in this particular case (as will be seen) the sphere is the simplest to work with mathematically.

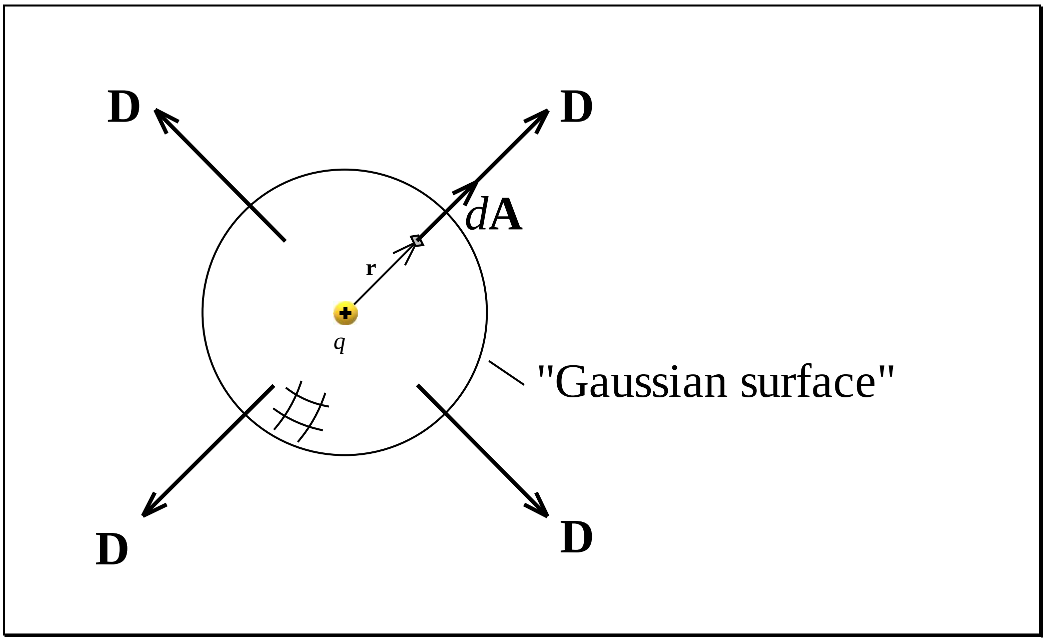

Figure 10.4 – Applying Gauss' Law to an isolated positive point charge

The flux density **D** is a *vector field* – it exists at all points in space. However, we are just interested in the flux density at a certain radial distance from the point charge, so we will just be finding **D** at all points on the surface of the sphere. If we want to find **D** at other radial distances, we can adjust the size of our sphere, i.e. the radius of our sphere is a variable.

In Figure 10.4, we have chosen to illustrate just four of the infinite number of **D** vectors which cover the surface of the sphere, and which, due to the spherical symmetry of the point charge, all point in a radial direction.

The vector **r** is a position vector which specifies the position of an arbitrary point on the surface of the sphere, with respect to the origin (where the point charge resides). The direction of **r**, for any particular point under consideration, is radially outward. The radius of the sphere is just the magnitude of this vector, r.

The vector **dA** is a differential area vector, directed radially outwards, which represents an infinitesimally small piece of the surface of the sphere. It has an infinitesimally small amount of electric flux, $d\Psi = \mathbf{D} \cdot \mathbf{dA}$, streaming through it.

We are seeking a formula for the electric flux density **D** at any point on the sphere, i.e. for any **r**. We start by stating Gauss' Law:

$$\oint \mathbf{D} \cdot \mathbf{dA} = q \tag{10.10}$$

Now since **D** and **dA**, for any point on the surface of the sphere, point radially outward, the dot product in the integral reduces to a simple expression. That is, we use the fact that:

$$\mathbf{D} \cdot \mathbf{dA} = D dA \cos 0° = D dA \tag{10.11}$$

to rewrite the integral as:

$$\oint D dA = q \tag{10.12}$$

Note that it was our choice of a sphere for the Gaussian surface that allowed us to get this simple expression. If we had chosen a different surface, such as a cube, we would need to embark on a lot of mathematical work to evaluate $\mathbf{D} \cdot \mathbf{dA}$ individually at every point on the surface of the cube, since **D** would be changing in both magnitude and direction at each point on the surface of the cube. Our choice of a sphere has "exploited" the symmetry of the **D** field.

Now we again invoke a symmetry argument – there is no way to orient a static sphere in space, i.e. there is no natural point of reference on the surface of a

sphere (this is true for a static sphere, which is what we have, but for a spinning sphere, such as the Earth, there is an axis of rotation which gives us two natural points on the surface – we call them the north and south pole). If there are no unique points on the surface of a sphere, then the *magnitude* (not the direction) of the flux density, D, must be a constant on the surface of the sphere. This argument again "exploits" the spherical symmetry of the point charge's field. If D is a constant, we can "bring it out the front" of our integral:

$$D\oint dA = q \tag{10.13}$$

Now $\oint dA$ indicates that we wish to find the sum of all the infinitesimally small areas dA over the whole of A, i.e. over the entire surface of the sphere. This must be just the surface area of the sphere:

$$\oint dA = A = 4\pi r^2 \tag{10.14}$$

Gauss' Law has now been reduced to:

$$DA = D4\pi r^2 = q \tag{10.15}$$

Rearranging, we now have a formula for the magnitude of the electric flux density:

$$D = \frac{q}{4\pi r^2} \tag{10.16}$$

Since we already know that the direction of **D** is radially outward, we can add back the direction information to get a *vector* formula for the electric flux density:

$$\boxed{\mathbf{D} = \frac{q}{4\pi r^2}\hat{\mathbf{r}} \quad \text{Cm}^{-2}} \tag{10.17}$$

10.5 Flux Density and Field Intensity

We already know the electric field intensity at any point around the isolated point charge, from Coulomb's Law:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad \mathrm{Vm}^{-1} \tag{10.18}$$

Comparing Eq. (10.17) with Eq. (10.18), we see that:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{10.19}$$

It turns out that Eq. (10.19) is completely general for any charge distribution in a vacuum (or air). It provides a link between electric flux density and electric field intensity, and hence to the force experienced by a charge.

10.6 A Recipe for Solutions of Guass' Law

(Determining flux densities due to symmetrical charge distributions)

1. Draw a picture of the charge (source) geometry. Sketch in **D** flux lines. You can justify your **D** lines from the first postulate of our fluid model, Eq. (10.1).

2. Choose a Gaussian surface that:

   (i) *intersects the field point* at which **D** is required. (If this condition is not met, the solution for **D** will not be relevant at this required point).

   (ii) *possesses the same symmetry as the charge geometry*. If a point source, then spherical symmetry; if a line, then cylindrical; if a sheet, then flat faces are required.

   (iii) *is* ***closed***. Gauss' Law is not valid for open surfaces. Hence ends and side faces should be added to the surfaces where necessary. These should be chosen with (iv) in mind.

(iv) *has each face either parallel or perpendicular to the sketched* **D** *lines*. Sketch in the Gaussian surface.

3. Write Gauss' Law $\oint \mathbf{D} \cdot \mathbf{dA} = q_{\text{enclosed}}$.

4. Evaluate the LHS as follows:

   a) Since integration of a vector is not possible, the **dot product must be reduced to a scalar** first. Hence separate the integral into a sum of integrals over surfaces that have, according to 2 (iv):

   - **dA** parallel to **D** lines: $\mathbf{D} \cdot \mathbf{dA} = DdA$
   - **dA** perpendicular to **D** lines: $\mathbf{D} \cdot \mathbf{dA} = 0$

   b) With the Gaussian surface chosen as in 2 (ii), the value of **D** should be constant over the remaining surface where $\mathbf{D} \cdot \mathbf{dA} = DdA$. If it is not, then a more general method (Laplace's or Poisson's equation) should be chosen for the problem. Justify *why D* is constant in your case.

   Hence $\int \mathbf{D} \cdot \mathbf{dA} = \int DdA = D\int dA$ since D is a constant.

   Sum the area $\int dA$. Arbitrary dimensions for the Gaussian surface can be chosen, since the final answer will necessarily involve their cancellation. The surface is simply a mathematical figment, and its dimensions cannot influence the physical properties.

5. Evaluate the RHS: Sum the charge enclosed within the arbitrary dimensions of the Gaussian surface. If the charge is not uniform, this may involve an integration.

6. Hence now solve the simple linear equation in D.

7. By reference to your sketch, state mathematically or in words the direction of the *vector* **D**. State the units (Cm^{-2}).

E and **F** can, if required, be now found from the expression for **D**.

10.7 Tubes of Electric Flux

The electric flux density **D** has been shown to be simply related to the electric field intensity **E** through the constant ε_0 in free space.

We have seen that lines of electric force in general do not cross (except where the electric field intensity is equal to zero). If we construct a tube of electric lines of force placed side by side as shown in the figure below we would expect that no flux would cross the walls of the tube, since the electric flux density **D** has the same direction as the electric field intensity **E** and to do so would require a **D** vector which intersected the wall of the tube.

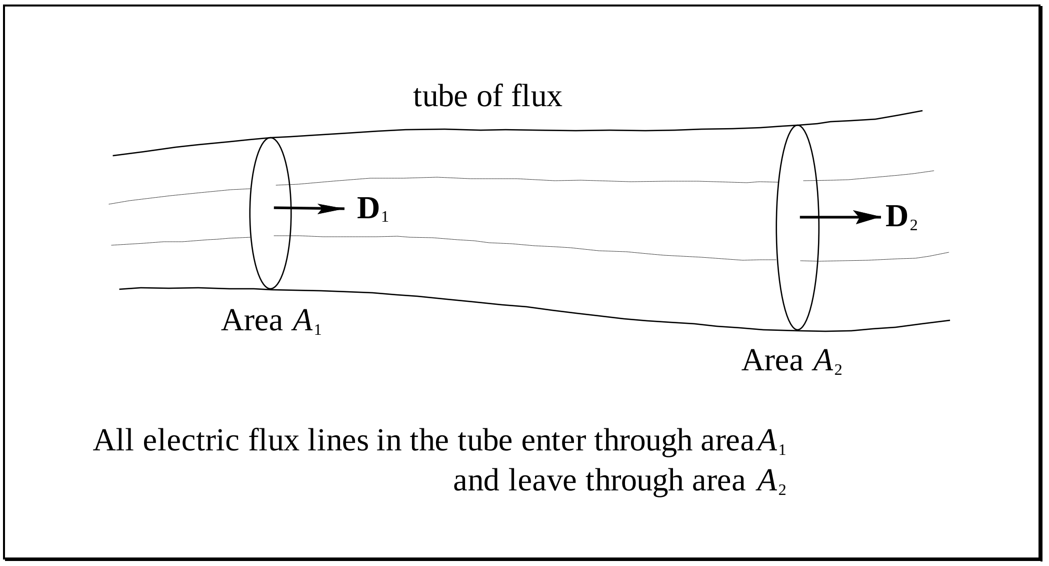

Figure 10.5 – Illustrating the concept of a tube of electric flux

From Figure 10.5, we may say that, for a tube of flux:

$$\psi_1 = \psi_2 \tag{10.20}$$

where ψ_1 is the flux entering the tube through area A_1 and ψ_2 is the flux leaving the tube through area A_2. If D is constant at every point on the surfaces A_1 and A_2 then:

$$D_1 A_1 = D_2 A_2 \tag{10.21}$$

It might be reasonable to call the lines of electric force therefore electric flux lines and the tube made up of the lines of electric force, a tube of electric flux.

10.8 Summary

- To model the effect of action-at-a-distance, we use the concept of an imaginary fluid called electric flux that streams throughout all of space. Positive charges are a source of electric flux and negative charges are a sink for electric flux. The source strength of the electric flux is equal to the amount of 'charge'.

- Area vectors have the magnitude of the area they represent, and a direction which is perpendicular (normal) to the surface they represent.

- The flux density at any point in space is denoted by the vector **D** which gives the amount of flux streaming through an infinitesimally small area at that point.

- Gauss' Law states that the total amount of flux streaming out of an enclosed surface is given by:

$$\oint \mathbf{D} \cdot \mathbf{dA} = q_{\text{enclosed}}$$

  where the "charge enclosed" is that captured by the volume created by the enclosing surface.

- There is a fundamental relationship between the electric flux density and the electric field intensity in a vacuum (or air), given by:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

- A tube of flux can be considered streaming through space, with its walls defined by lines of electric flux (and therefore electric lines of force). The flux entering (through area 1) the tube and leaving (through area 2) the tube must be the same, so that:

$$D_1 A_1 = D_2 A_2$$

10.9 References

Beard, G.: *Electrostatics and Magnetostatics*, NSWIT, 1985.

Bryce, P.: *Field Theory 2 Notes*, NSWIT, 1988.

Exercises

1.

Use Gauss' Law to obtain the electrostatic flux density **D** and hence the field intensity **E**, at a distance *r*, in a vacuum, from:

(a) the centre of a uniformly charged spherical shell, with radius *a*, and a total charge *q*, when $r \geq a$.

(b) as in (a) but with $r < a$.

(c) an infinite line charge with uniform charge density λ Cm^{-1}.

(d) an infinite plane with uniform charge density σ Cm^{-2}.

2.

In Q1 (c), if a second, initially uncharged, metallic plate were now brought into a position opposite the charged plane, how will the **E** field be affected:

(a) If the plate is unearthed.

(b) If the plate is earthed.

Explain.

3.

Using Gauss' Law, derive the value of the electrostatic **E** field within a hollow charged metallic sphere.

[This relates to the early experiment performed by Cavendish which led to the realisation that electric forces obeyed and “inverse-square law”.]