Lecture 5 – Transmission Line Parameters

Capacitance of overhead transmission lines. Inductance of overhead transmission lines. Calculation of geometric mean distances. Resistance of overhead transmission lines. Parameters of underground transmission lines.

Capacitance of Overhead Transmission Lines

Capacitances of a Multiconductor Transmission Line Running Parallel to an Earth Plane

(a) Line Charge

Assume an infinitely long horizontal line charge above a horizontal conducting "earth" plane.

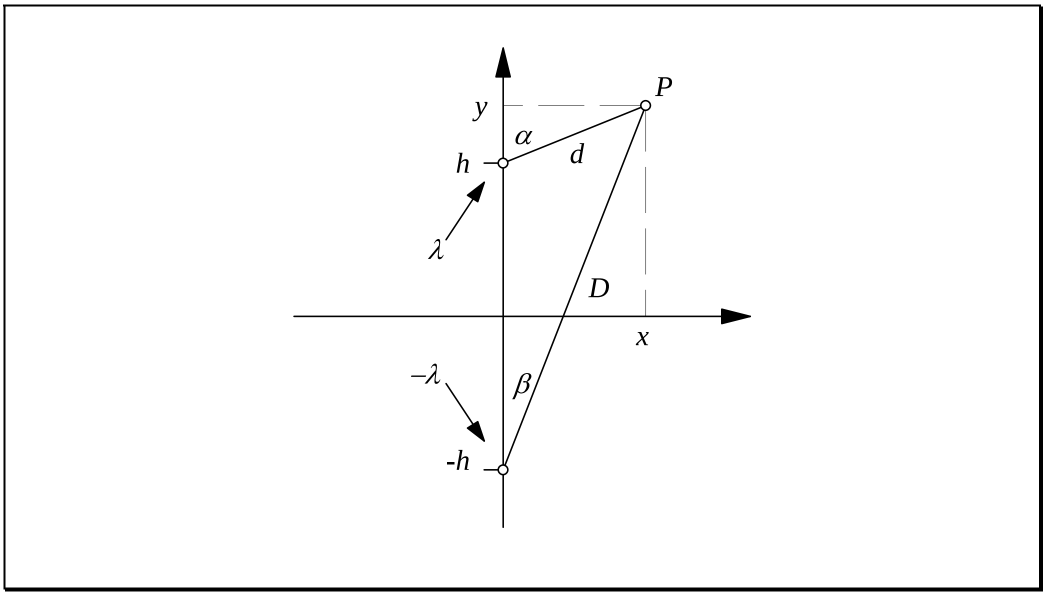


Figure 5.1 – Line charge above an earth plane

The earth plane (at $y=0$) is at zero potential. For determining the electric field in the space above the plane ($y>0$) we replace the plane with an image line charge.

5.2

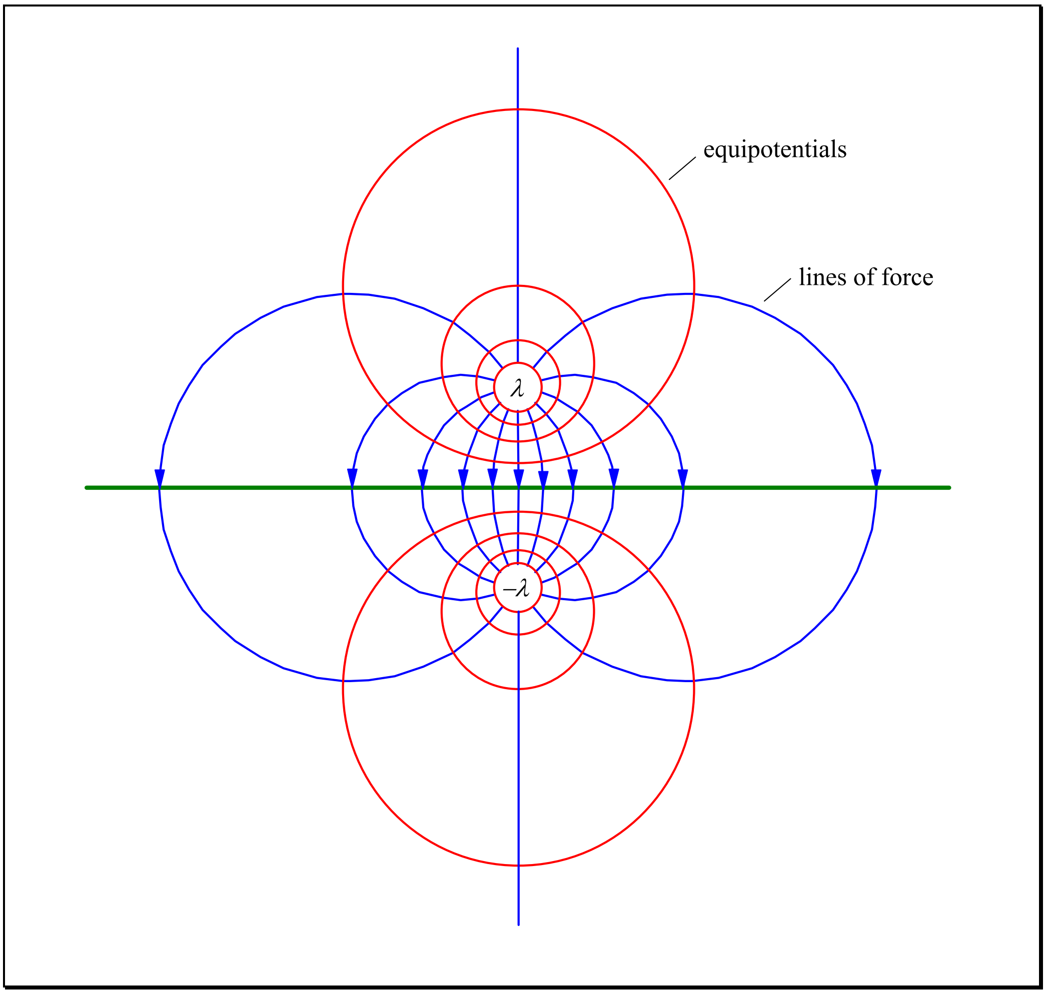


Figure 5.2 – Field plot of line charge with image

Let:

λ = line charge density in coulombs per metre (Cm^{-1}) (5.1)

h = height of the line charge above the earth plane (5.2)

By Gauss' Law, the electric field vector at the radial distance r from a line charge in free space is:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}} \tag{5.3}$$

To obtain potential V we integrate along path l:

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l} \tag{5.4}$$

Now set $V = 0$ at $r = h$ to obtain the potential at point P due to the positive charge alone:

$$V = -\int_h^d \frac{\lambda dr}{2\pi\varepsilon_0 r} \tag{5.5}$$

This is best illustrated by the diagram below:

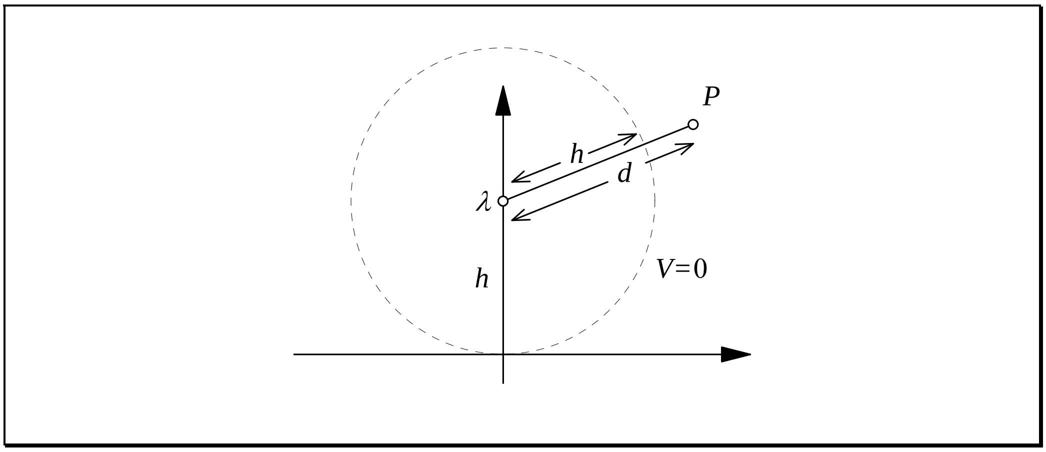


Figure 5.3 – Reference potential at distance *h*

Using superposition, we can add the potential due to the negative image charge to obtain the potential at point P with respect to the earth plane:

$$\begin{aligned} V &= -\int_h^d \frac{\lambda dr}{2\pi\varepsilon_0 r} - \int_h^D \frac{-\lambda dr}{2\pi\varepsilon_0 r} \\ &= \frac{\lambda}{2\pi\varepsilon_0}\left(-\ln d + \ln h + \ln D - \ln h\right) \\ &= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{D}{d} \end{aligned} \tag{5.6}$$

5.4

The electric field components in the *x* and *y* direction are:

$$E_x = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{d}\sin\alpha - \frac{1}{D}\sin\beta\right)$$
$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{d}\cos\alpha - \frac{1}{D}\cos\beta\right) \tag{5.7}$$

Hence:

$$E_x(x,y) = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{x}{x^2+(y-h)^2} - \frac{x}{x^2+(y+h)^2}\right)$$
$$E_y(x,y) = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{y-h}{x^2+(y-h)^2} - \frac{y+h}{x^2+(y+h)^2}\right) \tag{5.8}$$

At $y = 0$:

$$E_x(x,0) = 0$$
$$E_y(x,0) = \frac{\lambda}{\pi\varepsilon_0}\left(\frac{h}{x^2+h^2}\right) \tag{5.9}$$

$E_y(x,0)$ corresponds to an induced surface charge density ρ_s on the earth plane:

$$\rho_s = \varepsilon_0 E_y(x,0) = \frac{-\lambda}{\pi}\left(\frac{h}{x^2+h^2}\right) \quad \text{Cm}^{-2} \tag{5.10}$$

Integrating to obtain the total induced surface charge:

$$\begin{aligned}\int_{-\infty}^{\infty} \rho_s dx &= 2\int_{o}^{\infty} \rho_s dx \\ &= \frac{-2\lambda}{\pi}\int_{o}^{\infty} \frac{hdx}{x^2 + h^2} \\ &= \text{-}\lambda \quad \mathrm{Cm}^{-1} \end{aligned} \tag{5.11}$$

Thus the actual <u>induced surface charge per unit length</u> $(-\lambda)$ equals the <u>fictitious image charge per unit length</u>, and exactly balances the inducing line charge per unit length λ.

(b) Single Conductor

It can be shown that the equipotential surfaces near the line charge in **(a)** above are cylinders, and when the radius of the cylinder is much smaller than the distance h, the axis of the equipotential cylinder very nearly coincides with the line charge.

Hence the line charge model is valid for a long cylindrical conductor of radius R, and when $R << h$ the line charge and the conductor may be assumed to be coaxial. In these cases, applying Eq. (5.6) to a single isolated conductor we obtain the conductor potential:

$$V_{cond} = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{2h}{R} \tag{5.12}$$

The capacitance per unit length (conductor to earth) is then:

$$\boxed{C = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0}{\ln \dfrac{2h}{R}} \quad \text{Fm}^{-1}} \tag{5.13}$$

(c) Multiple Conductors

We can describe the charge / voltage relations of a set of *n* parallel conductors by an *n* x *n* matrix **P** of *Maxwell's potential coefficients*:

$$\boxed{\mathbf{V} = \mathbf{P}\boldsymbol{\lambda}} \tag{5.14}$$

where:

$$\mathbf{V} = \text{line voltage column vector (V)} \tag{5.15}$$

$$\boldsymbol{\lambda} = \text{charge density column vector (Cm}^{-1}\text{)} \tag{5.16}$$

From Eq. (5.6) we let:

$$\boxed{\mathbf{P} = \frac{1}{2\pi\varepsilon_0}\mathbf{A} \quad \mathrm{F}^{-1}\mathrm{m}} \tag{5.17}$$

where:

$$\boxed{\mathbf{A} = \text{matrix with elements } A_{ij} = \ln\frac{D_{ij}}{d_{ij}}} \tag{5.18}$$

(The conductor spacing is assumed to be large compared to the diameters).

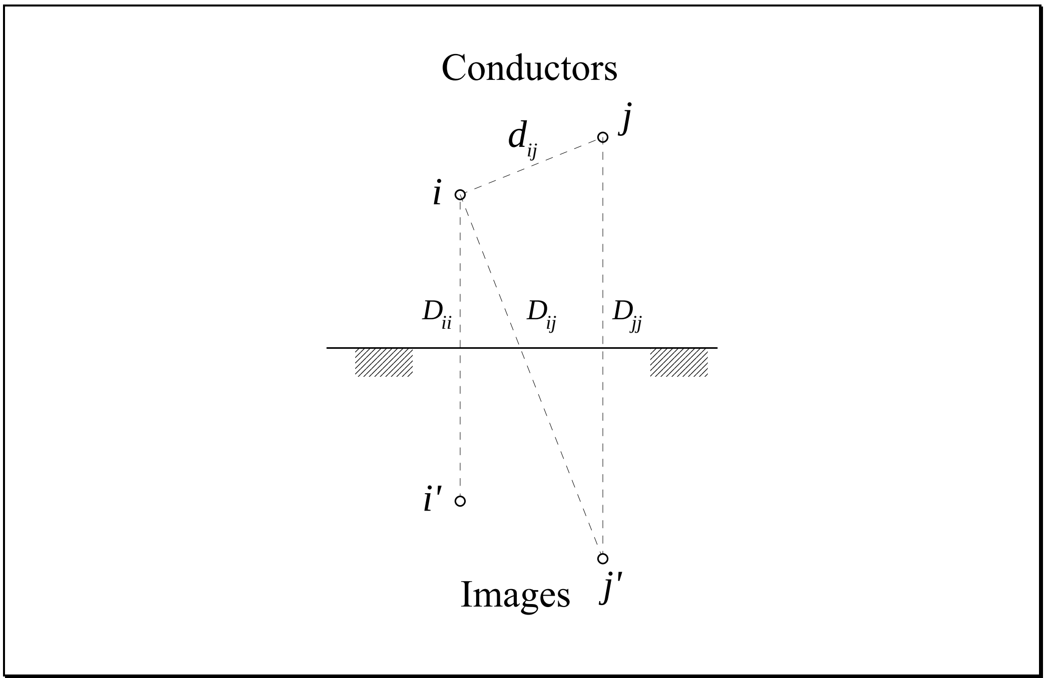


Figure 5.4 – Multiple conductors above an earth plane

For the diagonal elements $i = j$, and from Eq. (5.12):

$$d_{ii} = R_i = \text{radius of conductor } i \tag{5.19}$$

$$D_{ii} = 2h_i = \text{double height of conductor } i \tag{5.20}$$

The inverse of **P** is the capacitance per unit length matrix:

$$\boxed{\mathbf{C} = 2\pi\varepsilon_0 \mathbf{A}^{-1} \quad \mathrm{Fm}^{-1}} \tag{5.21}$$

Example

Two conductors of 10 mm diameter, conductor 1 is 5 m above ground, conductor 2 is 1 m higher and directly above conductor 1. Determine the capacitance matrix and an equivalent circuit of the conductor capacitances.

Solution:

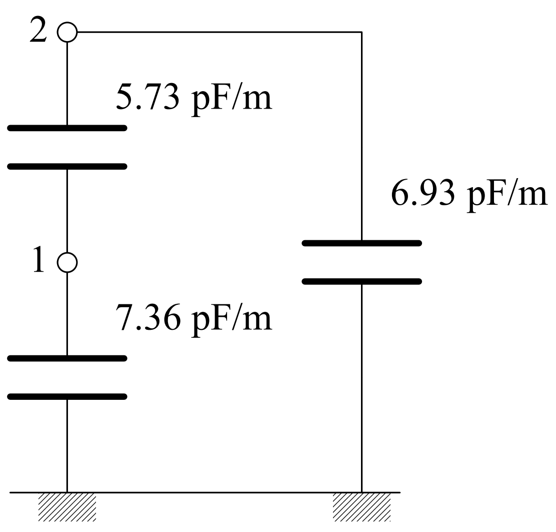
$$A_{11} = \ln\frac{10}{0.05} = 5.298 \quad A_{12} = \ln\frac{11}{1} = 2.398$$

$$A_{21} = A_{12} \qquad A_{22} = \ln\frac{12}{0.05} = 5.481$$

$$\mathbf{A} = \begin{bmatrix} 5.298 & 2.398 \\ 2.398 & 5.481 \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{23.288}\begin{bmatrix} 5.481 & -2.398 \\ -2.398 & 5.298 \end{bmatrix}$$

$$\begin{aligned} \mathbf{C} &= 2\pi \times 8.8542 \times 10^{-12} \mathbf{A}^{-1} \ \text{Fm}^{-1} \\ &= \begin{bmatrix} 13.09 & -5.73 \\ -5.73 & 12.66 \end{bmatrix} \text{pFm}^{-1} \end{aligned}$$

The equivalent network is shown below:



(d) Bundle Conductors

Bundle conductors are made up of a number of parallel, usually identical, sub-conductors. Bundle conductors may be used to increase the current carrying capacity, to lower the inductance, or to reduce the electric stress at the surface of the conductor. The last reason is the most common for high voltage transmission lines of 220 kV and above.

Consider a conductor a made up of m sub-conductors. All sub-conductors are at the same potential V_a, hence from Eq. (5.14):

$$\begin{bmatrix} V_a \\ V_a \\ \cdot \\ V_a \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdot & P_{1m} \\ P_{21} & P_{22} & \cdot & P_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ P_{m1} & \cdot & \cdot & P_m \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \lambda_m \end{bmatrix} \tag{5.22}$$

Assume the total charge density λ_a is divided equally between the m sub-conductors (a fair approximation in most cases). Then, from the first row, we obtain:

$$\begin{aligned} V_a &\approx \frac{\lambda_a}{m}\left(P_{11} + P_{12} + \cdots + P_{1m}\right) \\ &= \frac{\lambda_a}{2\pi\varepsilon_0 m} \sum_{j=1}^{m} \ln \frac{D_{1j}}{d_{1j}} \\ &= \frac{\lambda_a}{2\pi\varepsilon_0 m} \ln \frac{\prod_{j=1}^{m} D_{1j}}{\prod_{j=1}^{m} d_{1j}} = \frac{\lambda_a}{2\pi\varepsilon_0} \ln \frac{\left(\prod_{j=1}^{m} D_{1j}\right)^{\frac{1}{m}}}{\left(\prod_{j=1}^{m} d_{1j}\right)^{\frac{1}{m}}} \end{aligned} \tag{5.23}$$

We get slightly different values from the $m-1$ remaining rows, but taking all rows into the account we finally arrive at:

$$\boxed{V_a = \frac{\lambda_a}{2\pi\varepsilon_0}\ln\frac{D_{aa}}{d_{aa}}} \tag{5.24}$$

where:

$$D_{aa} = \left[\prod_{i=1}^{m}\left(\prod_{j=1}^{m} D_{ij}\right)\right]^{\frac{1}{m^2}} \tag{5.25}$$

$$d_{aa} = \left[\prod_{i=1}^{m}\left(\prod_{j=1}^{m} d_{ij}\right)\right]^{\frac{1}{m^2}} \tag{5.26}$$

D_{aa} and d_{aa} are geometric mean distances (GMD).

D_{aa} = mutual GMD between the bundle conductor *a* and its image

d_{aa} = self-GMD of the bundle conductor *a*

m = number of sub-conductors in the bundle

d_{aa} is also known as the "geometric mean radius" (GMR) of the conductor. The description "geometric mean radius" is mathematically incorrect, but it is the equivalent radius of a simple cylindrical conductor that would have the same capacitance as the bundle conductor.

Eq. (5.24) is identical to Eq. (5.12) with $2h$ and R replaced with corresponding geometric mean distances. We can extend the principle to the multiconductor transmission line in which each conductor consists of n parallel sub-conductors. Then the distances D_{ij} and d_{ij} in Eq. (5.18) become geometric mean distances as follows:

D_{ij} = mutual GMD between the bundle conductor i and the image of j

d_{ij} = mutual GMD between the bundle conductors i and j

Example

Two conductors, conductor 1 is 5 m above ground, conductor 2 is 1 m higher and directly above conductor 1, as in the previous example. Both conductors are connected in parallel. Determine the capacitance to earth.

Solution:

1) From the solution of the previous example:

$$C = 7.36 + 6.93 = 14.29 \text{ pFm}^{-1}$$

2) Treating the two conductors as a single bundle conductor:

$$d_{11} = d_{22} = 0.05 \text{ m} \qquad d_{12} = d_{21} = 1 \text{ m}$$

Equivalent radius $d_{aa} = (d_{11}d_{12}d_{21}d_{22})^{\frac{1}{4}} = (d_{11}d_{12})^{\frac{1}{2}} = \sqrt{0.05 \times 1} = 0.2236 \text{ m}$

$$D_{11} = 10 \text{ m} \qquad D_{22} = 12 \text{ m} \qquad D_{12} = D_{21} = 11 \text{ m}$$

$$D_{aa} = (D_{11}D_{12}D_{21}D_{22})^{\frac{1}{4}} = (10 \times 11^2 \times 12)^{\frac{1}{4}} = 10.98 \text{ m}$$

(almost the same as the centre line distance of 11 m)

$$C = \frac{2\pi \times 8.8542 \times 10^{-12}}{\ln \frac{10.98}{0.2236}} \text{ Fm}^{-1} = 14.29 \text{ pFm}^{-1}$$

Capacitances of a Three-Phase Overhead Transmission Line

We will assume a simple transmission line with just three conductors labelled *a*, *b* and *c*. We assume all conductors to be identical, except for their position in space. Image conductors at a depth equal to the conductor height are assumed.

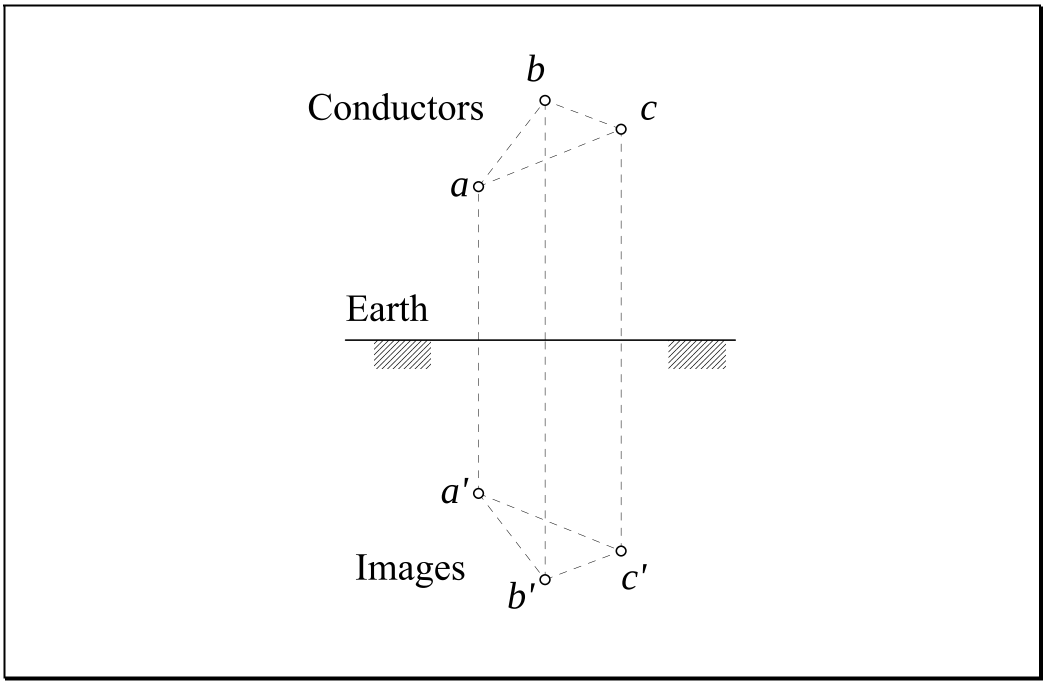


Figure 5.5 – Three-phase transmission line and its image

D_{ab}, D_{bc}, D_{ca} = mutual GMDs between conductor and image of another conductor

D_{aa}, D_{bb}, D_{cc} = mutual GMDs between conductor and its own image

d_{ab}, d_{bc}, d_{ca} = mutual GMDs between conductors

d_{aa}, d_{bb}, d_{cc} = self-GMD of conductor (radius of a cylindrical conductor)

For the mutual GMDs centre-line distances are *usually* sufficient as an approximation.

Ideally the conductors are transposed so that each conductor occupies all three positions in equal proportions over the length of the transmission line, as illustrated below:

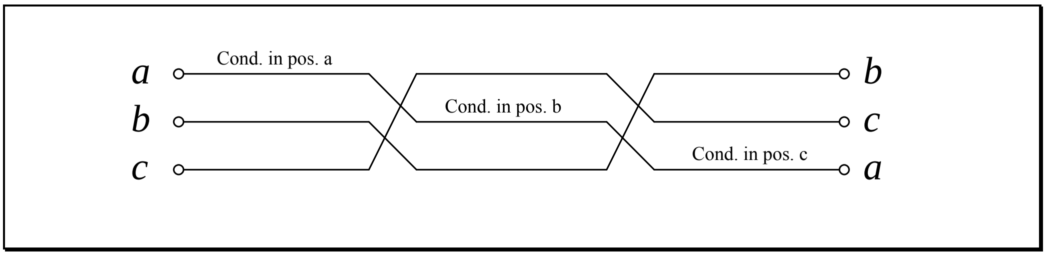


Figure 5.6 – A "barrel" of transpositions

For the first (LH) section of the "barrel" we obtain the potential coefficients from Eq. (5.17):

$$\mathbf{P} = \frac{1}{2\pi\varepsilon_0}\begin{bmatrix} A_{aa} & A_{ab} & A_{ac} \\ A_{ba} & A_{bb} & A_{bc} \\ A_{ca} & A_{cb} & A_{cc} \end{bmatrix} \text{ Fm}^{-1} \tag{5.27}$$

$$A_{ij} = A_{ji} = \ln\frac{D_{ij}}{d_{ij}} \tag{5.28}$$

The values in the matrix are rotated for the other two sections. Thus, for the complete transposed line:

$$\mathbf{P}_{abc} = \begin{bmatrix} P_s & P_m & P_m \\ P_m & P_s & P_m \\ P_m & P_m & P_s \end{bmatrix} \tag{5.29}$$

The diagonal and non-diagonal elements respectively are:

$$P_s = \frac{1}{2\pi\varepsilon_0}\frac{1}{3}\left(\ln\frac{D_{aa}}{d_{aa}} + \ln\frac{D_{bb}}{d_{aa}} + \ln\frac{D_{cc}}{d_{aa}} \right) = \frac{1}{2\pi\varepsilon_0}\ln\frac{\sqrt[3]{D_{aa}D_{bb}D_{cc}}}{d_{aa}} \tag{5.30}$$

$$P_m = \frac{1}{2\pi\varepsilon_0}\frac{1}{3}\left(\ln\frac{D_{ab}}{d_{ab}} + \ln\frac{D_{bc}}{d_{bc}} + \ln\frac{D_{ca}}{d_{ca}} \right) = \frac{1}{2\pi\varepsilon_0}\ln\frac{\sqrt[3]{D_{ab}D_{bc}D_{ca}}}{\sqrt[3]{d_{ab}d_{bc}d_{ca}}} \tag{5.31}$$

Applying the symmetrical component transformation, we get:

$$\mathbf{P}_{012} = \mathbf{H}^{-1}\mathbf{P}_{abc}\mathbf{H} = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & P_2 \end{bmatrix} \tag{5.32}$$

where:

$$\mathbf{P}_{012} = \mathbf{H}^{-1}\mathbf{P}_{abc}\mathbf{H} = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & P_2 \end{bmatrix} \tag{5.33}$$

and:

$$\begin{aligned} P_1 &= P_2 = P_s - P_m \\ P_0 &= P_s + 2P_m \end{aligned} \tag{5.34}$$

Now:

$$P_1 = P_2 = P_s - P_m$$
$$= \frac{1}{2\pi\varepsilon_0}\left(\ln\frac{\sqrt[3]{D_{aa}D_{bb}D_{cc}}}{d_{aa}} - \ln\frac{\sqrt[3]{D_{ab}D_{bc}D_{ca}}}{\sqrt[3]{d_{ab}d_{bc}d_{ca}}} \right) \tag{5.35}$$
$$= \frac{1}{2\pi\varepsilon_0}\left(\ln\frac{\sqrt[3]{d_{ab}d_{bc}d_{ca}}}{d_{aa}} - \ln\frac{\sqrt[3]{D_{ab}D_{bc}D_{ca}}}{\sqrt[3]{D_{aa}D_{bb}D_{cc}}} \right)$$

Inverting P_1 we obtain the positive (and negative) sequence capacitance per unit length:

$$\boxed{C_1 = C_2 = \frac{2\pi\varepsilon_0}{\ln\frac{d_{eq}}{d_{aa}} - \ln\frac{\sqrt[3]{D_{ab}D_{bc}D_{ca}}}{\sqrt[3]{D_{aa}D_{bb}D_{cc}}}}} \tag{5.36}$$

where:

$$d_{eq} = \sqrt[3]{d_{ab}d_{bc}d_{ca}} = \text{“equivalent spacing”} \tag{5.37}$$

When the conductor spacing is very small compared to the height from the ground, then $\ln\frac{\sqrt[3]{D_{ab}D_{bc}D_{ca}}}{\sqrt[3]{D_{aa}D_{bb}D_{cc}}}$ is very small, and the capacitance per unit length approaches:

$$\boxed{C_1 = C_2 \approx \frac{2\pi\varepsilon_0}{\ln\frac{d_{eq}}{d_{aa}}}} \tag{5.38}$$

We could find the zero sequence capacitance per unit length C_0 by using $P_0 = P_s + 2P_m$ but there is an easier way. The capacitance per unit length to earth of all three conductors in parallel $= 3C_0$, hence the zero sequence capacitance per unit length is:

$$\boxed{C_0 = \frac{2\pi\varepsilon_0}{3\ln\dfrac{D_{3c}}{d_{3c}}}} \tag{5.39}$$

where:

$$D_{3c} = \left(D_{aa}D_{bb}D_{cc}D_{ab}^2D_{bc}^2D_{ca}^2\right)^{\frac{1}{9}} \tag{5.40}$$

= mutual GMD between all three conductors and their images

and:

$$d_{3c} = \left(d_{aa}d_{bb}d_{cc}d_{ab}^2d_{bc}^2d_{ca}^2\right)^{\frac{1}{9}} = \sqrt[3]{d_{aa}d_{eq}^2} \tag{5.41}$$

= self-GMD of the three conductors in parallel

Inductance of Overhead Transmission Lines

Inductances of a Multiconductor Transmission Line Running Parallel to an Earth Plane

(a) Single Conductor Near a Perfectly Conducting Earth

Assume an infinitely long horizontal thin conductor at $y = h$ above a horizontal perfectly conducting "earth" plane at $y = 0$.

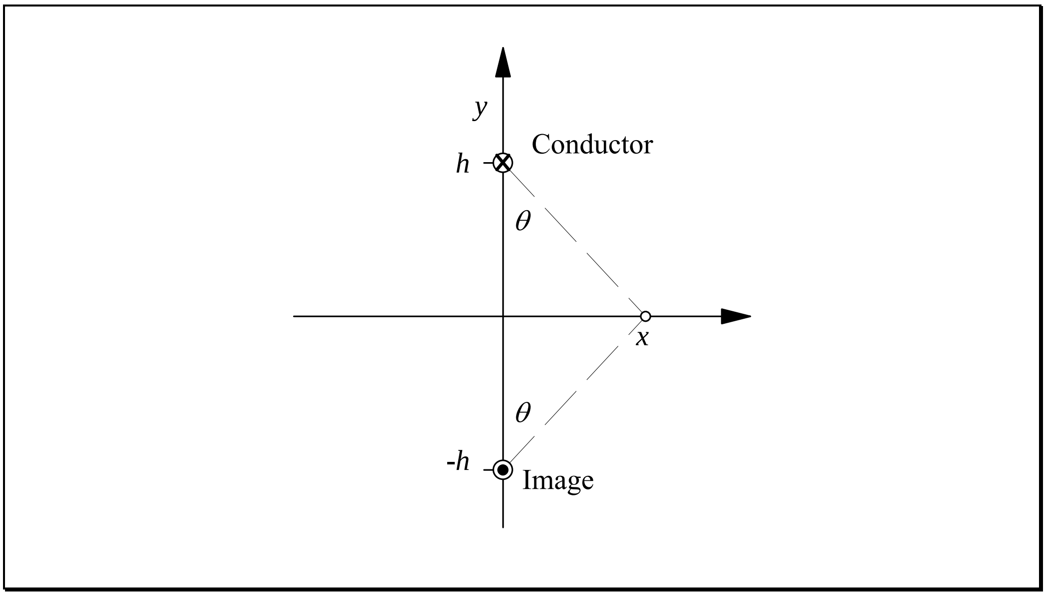


Figure 5.7 – Single conductor above an earth plane

Alternating flux cannot exist in a perfect conductor, suggesting an effective permeability of zero. Then we can, with some justification, try using an image current of equal magnitude and opposite direction. We replace the plane with the image current at $y = -h$, and calculate the resultant magnetic field in the region $y > 0$.

From Ampère's Law, the magnetic field vector at the radial distance *r* from the conductor is directed clockwise and has a magnitude:

$$H = \frac{I}{2\pi r} \tag{5.42}$$

At $y=0$ the horizontal and vertical components of **H**, due to the conductor and the image, are:

$$H_x = -\frac{2I}{2\pi r}\cos\theta = -\frac{Ih}{\pi\left(x^2+h^2\right)}, \quad H_y = 0 \tag{5.43}$$

H_x is actually discontinuous at $y=0$. Neither **B** nor **H** can exist inside the perfect earth conductor (whatever the permeability might actually be). The discontinuity has to be accounted for by a surface current density equal to H_x. The total surface current is:

$$\int_{-\infty}^{\infty} H_x dx = 2\int_0^{\infty} H_x dx = -\frac{2Ih}{\pi}\int_0^{\infty}\frac{dx}{\left(x^2+h^2\right)} = -I \tag{5.44}$$

Thus, there is a return current in the earth's surface under the conductor, and our rather dubious adoption of the method of images appears to be vindicated.

Flux linking the conductor per unit length is:

$$\phi = \frac{\mu_0 I}{2\pi}\int_R^h \frac{dr}{r} + \frac{\mu_0 I}{2\pi}\int_h^{2h}\frac{dr}{r} \tag{5.45}$$

where R = conductor radius (there is no flux inside a perfect conductor). Hence:

$$\begin{aligned}\phi &= \frac{\mu_0 I}{2\pi}\left(\ln h - \ln R + \ln 2h - \ln h\right) \\ &= \frac{\mu_0 I}{2\pi}\ln\frac{2h}{R} \quad \text{Wbm}^{-1}\end{aligned} \tag{5.46}$$

Since $L = \phi/I$, the inductance per unit length of the conductor with perfect earth return is:

$$\boxed{L = \frac{\mu_0}{2\pi} \ln \frac{2h}{R} \quad \text{Hm}^{-1}} \tag{5.47}$$

(b) Multiple Conductors Near a Perfectly Conducting Earth

Assume a second conductor, not carrying any current, at point *P*. The first conductor at $(0, h)$ carries current *I*.

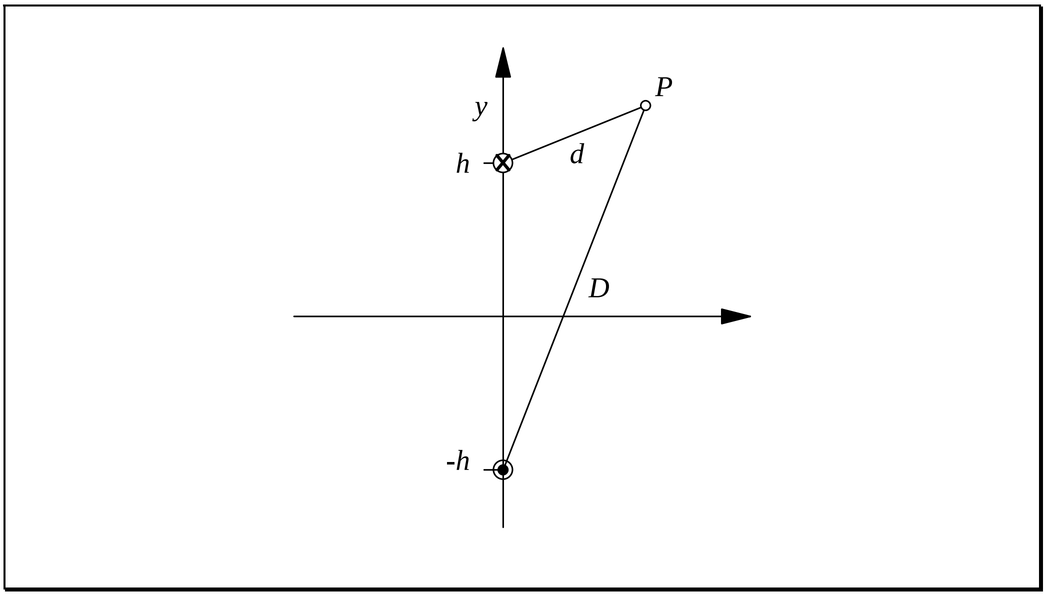


Figure 5.8 – Multiple conductors above an earth plane

The flux per unit length linking the second conductor is:

$$\phi_{21} = \frac{\mu_0 I}{2\pi}\int_d^h \frac{dr}{r} + \frac{\mu_0 I}{2\pi}\int_h^D \frac{dr}{r} \tag{5.48}$$

$$= \frac{\mu_0 I}{2\pi}\left(\ln h - \ln d + \ln D - \ln h\right)$$

$$= \frac{\mu_0 I}{2\pi}\ln\frac{D}{d} \quad \text{Wbm}^{-1}$$

The mutual inductance per unit length is then:

$$\boxed{L_{21} = \frac{\mu_0}{2\pi} \ln \frac{D}{d} \quad \text{Hm}^{-1}} \tag{5.49}$$

The similarity of form between the formulae for inductance per unit length and potential coefficients is obvious, and we can now set up the inductance per unit length matrix for the *ideal* multiconductor transmission line:

$$\boxed{\mathbf{L} = \frac{\mu_0}{2\pi} \mathbf{A} \quad \text{Hm}^{-1}} \tag{5.50}$$

where **A** is identical to the matrix used in the calculation of potential coefficients.

(c) Practical Conductors

With practical conductors the current is not confined to the surface, but diffuses exponentially some distance into the conductor. We define the "skin depth":

$$\boxed{\delta = \sqrt{\frac{2\rho}{\omega\mu}} \quad \text{m}} \tag{5.51}$$

where:

ρ = resistivity (Ωm) (5.52)

ω = angular frequency (rads^{-1}) (5.53)

μ = permeability ($\mu_r\mu_0$) of the conductor material (5.54)

($\mu_r = 1$ for copper and aluminium)

At 50 Hz and usual operating temperatures, the skin depth is about 10 mm for copper, and 13 mm for aluminium. Exact calculation of the current distribution in a cylindrical conductors requires Bessel functions, but for practical purposes we may assume uniform current distribution if the conductor radius is less than the skin depth.

The flux inside the conductor, only partially linked to the current, contributes an additional self-inductance to each conductor. Mutual inductances are not affected. It can be shown that for sold cylindrical conductors and uniform current distribution this "internal inductance" per unit length is a small fixed value equal to:

$$\boxed{L_{\text{internal}} = \frac{\mu_0}{8\pi} = 0.05 \quad \mu\text{Hm}^{-1}} \tag{5.55}$$

regardless of size (for a proof, refer to the prescribed textbook, or others).

From Eq. (5.50), the self-inductance per unit length is given by:

$$L_{ii} = \frac{\mu_0}{2\pi} \ln \frac{D_{ii}}{d_{ii}} \tag{5.56}$$

where we have $D_{ii} = 2h$ and $d_{ii} = R$ for an ideal conductor (zero skin depth). For the low frequency real conductor with uniform current distribution:

$$\begin{aligned} L_{ii} &= \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{D_{ii}}{d_{ii}} \\ &= \frac{\mu_0}{2\pi} \left(\frac{1}{4} + \ln \frac{D_{ii}}{R} \right) \\ &= \frac{\mu_0}{2\pi} \ln \left(\frac{D_{ii}}{R} e^{\frac{1}{4}} \right) \end{aligned} \tag{5.57}$$

Therefore Eq. (5.56) is also valid for the real conductor if we use:

$$d_{ii} = e^{-\frac{1}{4}} R = 0.778R \tag{5.58}$$

This equation gives the self-GMD of a solid cylindrical conductor with uniform current distribution.

☞ Note that there is a difference in the meaning of the GMDs used in inductance and capacitance calculations. In capacitance calculation we need the GMD between surface charge densities, while in inductance calculation we need the GMD between the current densities. The numerical difference is quite significant for the self-GMDs, but usually insignificant for mutual GMDs. Furthermore, with relatively large spacing between conductors the mutual GMDs may often be replaced with the corresponding centre-line distances.

(d) Real Earth

The problem of self and mutual impedances of parallel conductors with earth return was solved independently, and almost simultaneously, by Pollaczek and Carson in 1926. The finite resistivity of the earth causes the earth return currents (surface currents in an ideal earth) to penetrate well below the surface. The effect of this penetration is equivalent to increasing the image distances D_{ii} and D_{ij} and thereby increase all inductances (self and mutual). At the low frequencies and typical conductor heights of power lines the effective values of D_{ii} and D_{ij} are much larger than double conductor height and approach the "equivalent depth of earth return" D_E .

The self and mutual inductances per unit length of the conductors thus become:

$$\boxed{L_{ij} = \frac{\mu_0}{2\pi} \ln \frac{D_E}{d_{ij}} \quad \text{Hm}^{-1}} \tag{5.59}$$

where:

$$D_E = 659\sqrt{\frac{\rho}{f}} \quad \text{(m)} \tag{5.60}$$

ρ = resistivity of earth (Ωm) (5.61)

f = frequency (Hz) (5.62)

d_{ii} = self-GMD of current distribution of conductor *i* (5.63)

d_{ij} = mutual GMD between conductors *i* and *j* (5.64)

Eq. (5.60) is valid only at low frequencies. The proof is too difficult to be included here. The equivalent depth of earth return D_E is directly related to the skin depth δ:

$$D_E = 1.31\delta \tag{5.65}$$

Values of D_E

Values of Earth Resistivity		**Depth of Earth Return**
Typical	10 to 1000 Ωm	$D_E \approx 300$ to 3000 m
Extremes:	Rock, up to 10 000 Ωm	$D_E \approx 9000$ m
	Sea water, down to 0.25 Ωm	$D_E \approx 47$ m

Typical values of D_E are seen to be much larger than the ideal earth image distances ($\approx 2h$).

Inductances of a Three-Phase Overhead Transmission Line

Assume we are dealing with real earth, so that the image conductors may be placed at the equivalent depth of earth return D_E. Also assume line height and all lateral dimensions are $<< D_E$, al three conductors are identical, and the line is transposed.

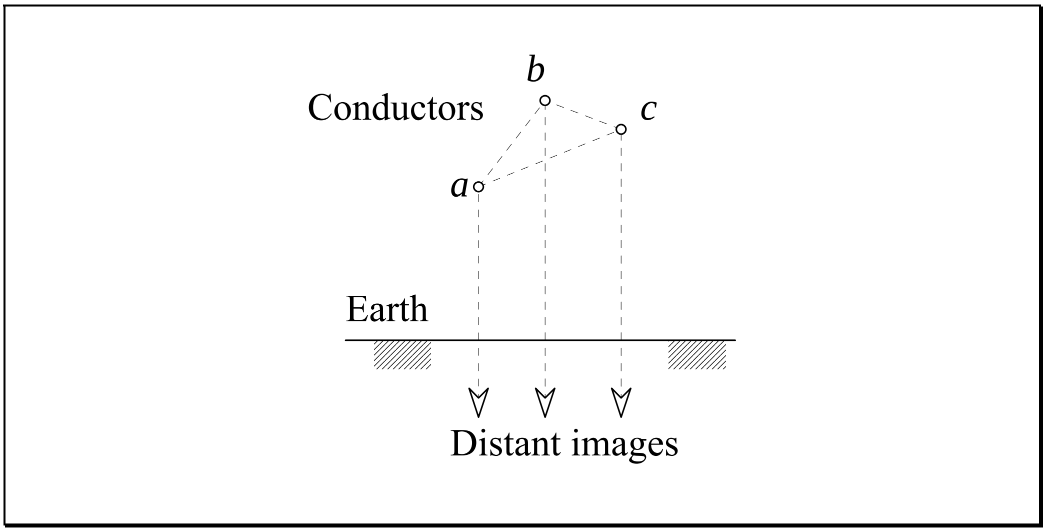


Figure 5.9 – Three-phase transmission line and its image

Then the self-inductance per unit length of each conductor is:

$$\boxed{L_s = \frac{\mu_0}{2\pi} \ln \frac{D_E}{d_{aa}} \quad \text{Hm}^{-1}} \tag{5.66}$$

The three equal mutual inductances per unit length are:

$$\begin{aligned} L_m &= \frac{\mu_0}{2\pi} \frac{1}{3} \left(\ln \frac{D_E}{d_{ab}} + \ln \frac{D_E}{d_{bc}} + \ln \frac{D_E}{d_{ca}} \right) \\ &= \frac{\mu_0}{2\pi} \ln \frac{D_E}{\sqrt[3]{d_{ab} d_{bc} d_{ca}}} \\ &= \frac{\mu_0}{2\pi} \ln \frac{D_E}{d_{eq}} \end{aligned} \tag{5.67}$$

where d_{eq} = "equivalent spacing" previously defined in Eq. (5.37).

5.28

The positive, negative and zero sequence inductances per unit length are:

$$L_1 = L_2 = L_s - L_m$$
$$L_0 = L_s + 2L_m \tag{5.68}$$

Hence:

$$\boxed{L_1 = L_2 = \frac{\mu_0}{2\pi}\ln\frac{d_{eq}}{d_{aa}} \quad \text{Hm}^{-1}} \tag{5.69}$$

and:

$$L_0 = \frac{\mu_0}{2\pi}\left(\ln\frac{D_E}{d_{aa}} + 2\ln\ln\frac{D_E}{d_{eq}}\right)$$
$$= \frac{\mu_0}{2\pi}\ln\frac{D_E^3}{d_{aa}d_{eq}^2} = 3\frac{\mu_0}{2\pi}\ln\frac{D_E}{\sqrt[3]{d_{aa}d_{eq}^2}} \tag{5.70}$$

Hence:

$$\boxed{L_0 = 3\frac{\mu_0}{2\pi}\ln\frac{D_E}{d_{3c}} \quad \text{Hm}^{-1}} \tag{5.71}$$

where

$$d_{3c} = \left(d_{aa}d_{bb}d_{cc}d_{ab}^2d_{bc}^2d_{ca}^2\right)^{\frac{1}{9}} = \sqrt[3]{d_{aa}d_{eq}^2} \tag{5.72}$$

= self-GMD of the three conductors in parallel

The d_{3c} used in Eq. (5.72) is formally similar to the corresponding d_{3c} used for capacitance calculation in Eq. (5.39), but the numerical values are different because different values of d_{aa} are used.

Calculation of Geometric Mean Distances

The concept of geometric mean distance (GMD) is intimately connected with the calculation of transmission line capacitances and inductances.

Generally, the geometric mean of n values of x is:

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \tag{5.73}$$

The "equivalent spacing" of three-phase transmission line conductors:

$$d_{eq} = \sqrt[3]{d_{ab} d_{bc} d_{ca}} = \text{"equivalent spacing"} \tag{5.74}$$

is a GMD of the above simple type.

We can evaluate the self-GMD of the cross-section area of any shape conductor (with uniform current density) by dividing the area into n equal elements (sub-areas), and calculate the geometric mean of the n^2 possible distances. The required GMD is then the limit value when $n \to \infty$.

We can calculate the mutual GMD between two cross-section areas by dividing the first area into n elements, and the second areas into m elements, then calculate the geometric mean of the $n \times m$ possible distances. The required GMD is then the limit value when both $n \to \infty$ and $m \to \infty$.

The mutual GMD between two areas with circular symmetry is their centre-line spacing.

Previous equations Eq. (5.25) and Eq. (5.26) cover the calculation of the self- and mutual GMDs of bundle conductors.

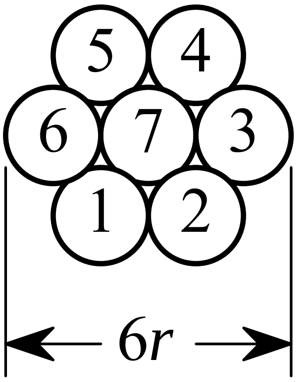
Examples of Self-GMD

Geometry	**Self-GMD**
Circular contour, radius R	R
Circular area, radius R	$e^{-\frac{1}{4}}R$
Rectangular area $a \times b$	$\approx 0.2235(a+b)$

Note of caution: The above concept of GMDs breaks down when there is a significant proximity effect, or a significant skin effect. Proximity effect causes both current and surface charge densities to be non-uniform. Skin effect causes non-uniform current density in conductors of large cross-section.

Example

Calculate the self-GMD of the cross-section of a stranded conductor composed of seven wires of radius r.



Distances in multiples of strand radius r:

$$
\begin{array}{ll}
d_{11} = d_{22} = \ldots = e^{-\frac{1}{4}} = a & \text{7 cases} \\
d_{12} = d_{23} = \ldots = 2 & \text{24 cases} \\
d_{13} = d_{24} = \ldots = 2\sqrt{3} = b & \text{12 cases} \\
d_{14} = d_{25} = \ldots = 4 & \text{6 cases}
\end{array}
$$

$$
\begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & a & 2 & b & 4 & b & 2 & 2 \\
2 & 2 & a & 2 & b & 4 & b & 2 \\
3 & b & 2 & a & 2 & b & 4 & 2 \\
4 & 4 & b & 2 & a & 2 & b & 2 \\
5 & b & 4 & b & 2 & a & 2 & 2 \\
6 & 2 & b & 4 & b & 2 & a & 2 \\
7 & 2 & 2 & 2 & 2 & 2 & 2 & a \\
\hline
\end{array}
$$

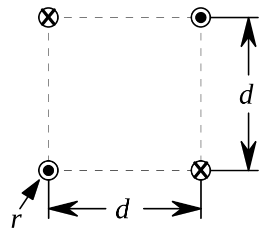
Distance matrix

Answer:

$$
\begin{aligned}
&= r\left(a^7 2^{24} b^{12} 4^6\right)^{\frac{1}{49}} \\
&= r\left(e^{-\frac{7}{4}} 2^{24} 2^{12} 3^6 2^{12}\right)^{\frac{1}{49}} \\
&= r e^{-\frac{1}{28}} 2^{\frac{48}{49}} 3^{\frac{6}{49}} \\
&= 2.177r
\end{aligned}
$$

Example

Calculate the capacitance and inductance per metre of a single-phase 50 Hz transmission line using four copper conductors, 10 mm diameter, arranged at the corners of a 500 mm square with polarities as shown.



$r = 5\ \text{mm} \quad d = 500\ \text{mm}$

For a simple 2-conductor transmission line remote from the earth:

$$C = \frac{\pi\varepsilon_0}{\ln\frac{D}{R}} \qquad\qquad L = \frac{\mu_0}{\pi}\ln\frac{D}{d_s}$$

where:

R = radius of conductor

D = distance between conductors

$d_s = e^{-\frac{1}{4}}R$

(Prove the above formula using Eq. (5.13), Eq. (5.56) and Eq. (5.58))

Conceptually we have only two *conductors*, each made up of two *sub-conductors*. Subconductor spacing is $\sqrt{2}d$.

Geometric mean *conductor* spacing is $D = \left(d^2 d^2\right)^{\frac{1}{4}} = d = 500\ \text{mm}$.

For inductance:

$$d_s = \left[\left(e^{-\frac{1}{4}}R\right)^2\left(\sqrt{2}d\right)^2\right]^{\frac{1}{4}} = e^{-\frac{1}{8}}2^{\frac{1}{4}}\sqrt{Rd} = 1.0495\sqrt{Rd} = 52.47 \text{ mm}$$

For capacitance:

$$d_s = \left[R^2\left(\sqrt{2}d\right)^2\right]^{\frac{1}{4}} = 2^{\frac{1}{4}}\sqrt{Rd} = 1.1892\sqrt{Rd} = 59.46 \text{ mm}$$

Then:

$$C = \frac{8.8542\pi}{\ln\frac{500}{59.46}} = 13.06 \text{ pFm}^{-1}$$

$$L = \frac{4\pi\times10^{-7}}{\pi}\ln\frac{500}{52.47} = 902 \text{ nHm}^{-1}$$

Example

Calculate the positive and zero sequence inductances per metre of a three-phase 50 Hz transmission line with conductors spaced at 2.7 m in a flat horizontal arrangement. Each conductor consists of seven strands of 3 mm diameter copper wire. The equivalent depth of earth return is given as 1300 m.

$$d_{eq} = \sqrt[3]{2.7 \times 2.7 \times 5.4} = 3.402 \text{ m}$$

From the result of the first GMD example:

$$d_{aa} = 2.177 \times 1.5 = 3.266 \text{ mm}$$

Then:

$$L_1 = 2 \times 10^{-7} \ln \frac{3402}{3.266} = 1.39 \ \mu\text{Hm}^{-1}$$

We also have:

$$d_{3c} = \sqrt[3]{d_{aa} d_{eq}^2} = \sqrt[3]{3.266 \times 3.402^2} = 0.3356 \text{ m}$$

Then:

$$L_0 = 3 \frac{\mu_0}{2\pi} \ln \frac{D_E}{d_{3c}} = 3 \times 2 \times 10^{-7} \ln \frac{1300}{0.3356} = 4.957 \ \mu\text{Hm}^{-1}$$

Resistance of Overhead Transmission Lines

Resistances of a Multiconductor Transmission Line Near the Surface of Earth

Apart from conductor resistance the earth return resistance needs to be considered. Assume a general *n*-conductor transmission line near a real earth surface. As with capacitance and inductance, we can set up an $n \times n$ matrix of loop resistances. The earth return resistance appears in all elements of the matrix, but the conductor resistances only in the diagonal elements.

At low frequencies this *earth return resistance* has been shown to be:

$$\boxed{\begin{aligned} R_E &= \frac{\omega\mu_0}{8} = \pi^2 f \times 10^{-7} \quad \Omega\text{m}^{-1} \\ &= 49.3\ \text{m}\Omega\text{m}^{-1} \qquad \text{at 50 Hz} \end{aligned}} \tag{5.75}$$

Eq. (5.75) is a low frequency approximation valid over the range in which the equivalent depth of earth return is much greater than the height or lateral spread of the conductors. The surprising feature is that the earth return resistance is independent of the resistivity of the earth! The inductance depends on the resistivity, but the resistance does not. The qualitative explanation is: the skin depth increases proportional to the square root of the resistivity, and so does the lateral spread of the earth return current. Therefore the effective cross-section of the earth return path increases in proportion with the resistivity, and the resistance remains constant. At high frequencies however the lateral spread is fixed, with the net result that the earth return resistance is proportional to the square root of frequency at high frequencies.

The resistance matrix has diagonal elements:

$$R_{ii} = R_E + R_i \tag{5.76}$$

and non-diagonal elements:

$$R_{ij} = R_E \tag{5.77}$$

where:

$$R_i \text{ = internal resistance per unit length of conductor } i \tag{5.78}$$

The Resistances of a Three-Phase Power Transmission Line

A transmission line can have both series and shunt components of resistance. For the purpose of network analysis the shunt components (insulator leakage, corona, etc.) can usually be neglected. The series components however are significant.

The effective series resistance must account for all losses of real power that are proportional to the square of the current. Thus power dissipated by induced currents in earth, earth wires, structural steelwork, etc. as well as power dissipated in the conductor itself would have to be accounted for in the effective series resistance. In case of practical power lines carrying only positive or negative sequence currents these stray effects are generally negligible, but not necessarily so in case of zero sequence currents.

The earth return resistance R_E cancels out for positive and negative sequences. Thus the positive sequence resistance is near enough just the conductor resistance, allowing for skin effect when necessary.

The sequence resistance are:

$$\boxed{\begin{aligned} R_1 &= R_a \\ R_0 &= R_a + 3R_E \end{aligned}} \tag{5.79}$$

where R_a = conductor (internal) resistance per metre, and R_E is defined in Eq. (5.75).

Parameters of Underground Transmission Lines

Without going into details of the design of various types of underground power cable, it suffices to say at this point that the inductances are smaller, roughly 25% to 50% of the inductance of an overhead transmission line. The capacitances of the underground cable are some 25 to 70 times higher than overhead line.

Summary

- The "method of images" is used to determine the capacitance and inductance per unit length of infinitely long overhead transmission lines. This method assumes an ideal earth plane.

- The formulae for inductance and capacitance per unit length utilise the concept of a geometric mean distance (GMD). The GMD can handle the cases of bundle conductors and multiple circuits (e.g. three-phase).

- Inductance calculations need to take into account skin depth and the finite resistance of the earth, giving rise to the concept of "equivalent depth of earth return". In addition, there is a fixed contribution to the overall inductance by the "internal inductance". These complications are not present for capacitance calculations.

- Resistance calculations need to take into account all loss mechanisms of the transmission line, not just the losses due to conduction.

References

Carmo, J.: *Power Circuit Theory Notes*, UTS, 1994.

Truupold, E.: *Power Circuit Theory Notes*, UTS, 1993.

Exercises

1.

Calculate the capacitance per metre of a long horizontal conductor suspended at a height of 10 m above the earth's surface for the following options:

(a) A cylindrical conductor, 25 mm diameter.

(b) A cylindrical conductor, 500 mm diameter.

(c) A bundle conductor comprising six 25 mm diameter cylindrical sub-conductors with their centres spaced evenly on a 500 mm diameter circle.

2.

A double-circuit three-phase transmission line is constructed so that its six conductors are placed at the corners of a regular hexagon with side length of two metres. The conductors are cylindrical and 20 mm in diameter. There are no transpositions.

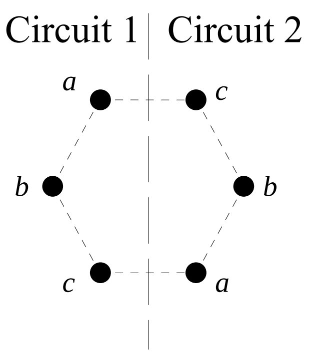


Figure A

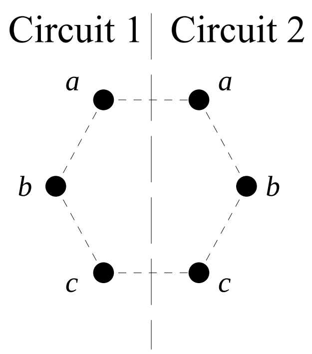


Figure B

(a) Calculate the line inductance (positive sequence) of circuit 1 with circuit 2 open-circuited.

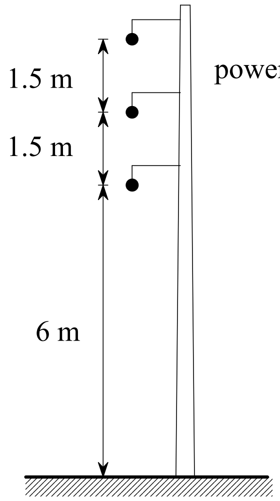
(b) Calculate the inductance of the two circuits in parallel as in Figure A.

(c) Calculate the inductance of the two circuits in parallel as in Figure B.

(d) Figure A is the better arrangement. Why? (There are two reasons.)

3.

A 132 kV three-phase urban transmission line has conductors arranged as shown. The line is transposed, and the conductors have a radius of 10 mm.



Determine the positive sequence inductance and capacitance per metre.

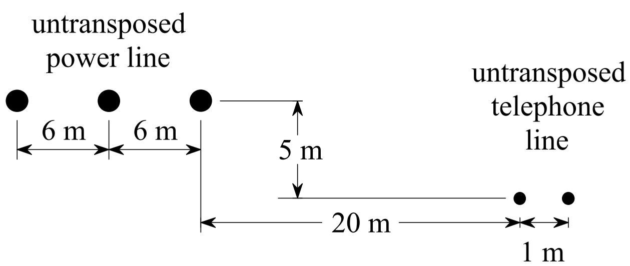
Can the proximity of the earth be ignored?

4.

An earth fault occurs on a three-phase transmission line. The fault current is 5000 A, the frequency is 50 Hz. The line is not provided with any earth wires. A wire fence runs for a distance of 1 km parallel to and 30 m from the line. Calculate the longitudinal induced voltage in the fence, assuming the earth to have a uniform resistivity of 250 Ωm. Is the answer realistic?

5.

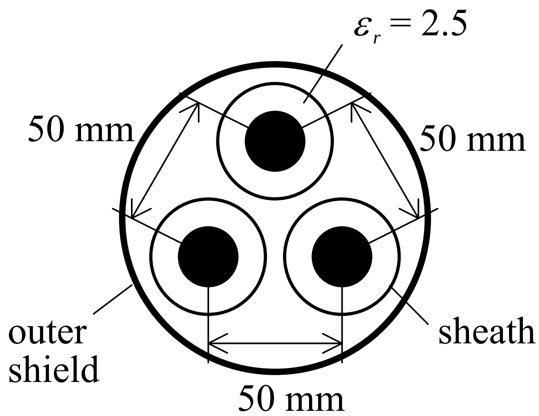
A telephone line runs parallel to a three-phase power line carrying 500 A symmetrical currents. The frequency is 50 Hz. Neither line is transposed. Relevant dimensions are as shown:



Calculate the voltage induced into the telephone line (loop voltage).

6.

An underground three-phase high voltage cable consists of three conductors, each 12 mm diameter. Each conductor is surrounded with a coaxial metallic sheath, with a 24 mm inside diameter. The insulation between the conductor and the sheath has a relative permittivity of 2.5. The centre-line distances between all three conductors are 50 mm. The sheaths are connected to earth, and are cross-bonded (transposed) so that no positive sequence currents are induced in them.



Calculate the positive sequence capacitance and inductance per metre.